SOME ESTIMATION METHODS FOR A NEW WEIGHTED LINDLEY DISTRIBUTION

Arbër Qoshja

University "AleksandërMoisiu", Durrës, Albania

e-mail: qoshjaa@gmail.com

Abstract: The weighted distributions provide a comprehensive understanding by adding flexibility in the existing standard distributions. In this article, we considered the new weighted Lindley distribution which belongs to the class of the weighted distributions. A new weighted Lindley distribution is proposed in 2015 by Asgharzadeh et al. which is a new generalization that provides better fits than the Lindley distribution and all of its known generalizations. In this paper, our main contribution is to compare the performances of the proposed methods of estimation for a two-parameter new weighted Lindley distribution by using Monte Carlo simulation. First we briefly describe different methods of estimations, namely maximum likelihood estimators, moments estimators, L-moment estimators, percentile based estimators and least squares estimators, and compare them using extensive numerical simulations. A real-life application is also presented for the illustration purpose.

Key words: *Weighted distribution; Lindley distribution; Bias; Mean squared errors; Simulations.*

1. Introduction

Weighted distributions are useful for better understanding of standard distributions and can extend distributions by adding flexibility. Also, truncated and damaged observations can be analyzed using weighted distributions. Azzalini (1985) proposed a new method for introducing a skewness parameter to the normal distribution based on a weighted function and obtained the skew-normal distribution. Lifetime distribution represents an attempt to describe, mathematically, the length of the life of a system or a device. Lifetime distributions are most frequently used in the fields like medicine, engineering etc. Many parametric models such as exponential, gamma, Weibull have been frequently used in statistical literature to analyze lifetime data. But there is no clear motivation for the gamma and Weibull distributions. They only have more general mathematical closed form than the exponential distribution with one additional parameter.

Recently, one parameter Lindley distribution has attracted the researchers for its use in modelling lifetime data, and it has been observed in several papers that this distribution has performed excellently. The Lindley distribution was originally proposed by Lindley in the context of Bayesian statistics, as a counter example of fudicial statistics which can be seen that as a mixture of $\exp(\theta)$ and

gamma(2, θ).
Some of the advances in the literature of Lindley distribution are given by Ghitany et al. (2011) who has introduced a two-parameter weighted Lindley distribution and has pointed that Lindley distribution is particularly useful in modeling biological data from mortality studies. Mahmoudi et al. (2010) have proposed generalized Poisson Lindley distribution. Bakouch et al. (2012) have come up with extended Lindley (EL) distribution, Adamidis and Loukas (1998) have introduced exponential geometric (EG) distribution. Shanker et al. (2013) have introduced a two-parameter Lindley distribution. Zakerzadeh et al. (2012) have proposed a new two parameter lifetime distribution: model and properties. M.K. Hassan (2008) has introduced convolution of Lindley distribution. Ghitany et al. (2013) worked on the estimation of the reliability of a stress-strength system from power Lindley distribution. Elbatal et al. (2013) has proposed a new generalized Lindley distribution.

Definition 1. A random variable X is said to have Lindley distribution with parameter θ if its probability density function is defined as:

$$g_X(x;\theta)=\frac{\theta^2}{(\theta+1)}(1+x)e^{-\theta x}; x>0,\theta>0 \quad (1)$$

with cumulative distribution function

$$G(x)=1-\frac{e^{-\theta x}(1+\theta+\theta x)}{1+\theta} \quad (2)$$

In 2015, Asgharzadeh et al. introduced a new two-parameters distribution called weighted Lindley distribution.[For more detail see: SAsgharzadeha, A., Bakouchb, H. S., Nadarajahc, S., & Sharafia, F.(2015). A new weighted Lindley distribution with application. Brazilian Journal of Probability and Statistics, 2015.]

Definition 2. A continuous random variable X is said to have new weighted Lindley distribution if the probability density of X is:

$$f(x)=\frac{\lambda^2(1+\alpha)^2}{\alpha\lambda(1+\alpha)+\alpha(2+\alpha)}(1+x)\left(1-e^{-\lambda\alpha x}\right)e^{-\lambda x}, \quad (3)$$

$$\lambda>0,\alpha>0,x>0.$$

The comulative distribution function of new weighted Lindley distribution is:

$$F(x)=-\frac{e^{-\lambda x}\{(1+\alpha)^2(1+\lambda+\lambda x)-[\lambda(1+\alpha)(1+x)+1]e^{-\alpha\lambda x}\}}{\alpha\lambda(1+\lambda)+\alpha(2+\alpha)} \quad (4)$$

The survival function of new weighted Lindley distribution is:

$$S(x) = \frac{e^{-\lambda x}\left\{(1+\alpha)^2(1+\lambda+\lambda x)-[\lambda(1+\alpha)(1+x)+1]e^{-\alpha\lambda x}\right\}}{\alpha\lambda(1+\lambda)+\alpha(2+\alpha)} \quad (5)$$

2. Maximum likelihood estimates

Fdistribution (3). Then, the likelihood function based on observed sample is defined as

$$\ell(\sim x,\lambda,\alpha) = \frac{\lambda^{2n}(1+\alpha)^{2n}}{[\alpha\lambda(1+\alpha)+\alpha(2+\alpha)]^n} e^{-\lambda\sum_{i=1}^{n} x_i} \prod_{i=1}^{n}\left[(1+x_i)(1-e^{-\lambda\alpha x_i})\right] \quad (6)$$

The log-likelihood function corresponding to (6), is given by

$$\log\ell = 2n\ln\lambda + 2n\ln(1+\alpha) - n\ln(\alpha) - n\ln[\lambda(1+\alpha)+2+\alpha] - \lambda\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\ln(1+x_i) + \sum_{i=1}^{n}\ln(1-e^{-\lambda\alpha x_i})$$

The maximum likelihood estimates $\bar{\lambda}$ of λ and $\bar{\alpha}$ of αcan be obtained as the simultaneous solutions of the following non-linear equations:

$$\frac{\partial\, log\ \ell}{\partial\alpha} = \frac{2n}{1+\alpha} - \frac{n[\lambda(1+2\alpha)+2(1+\alpha)]}{\alpha[\lambda(1+\alpha)+2+\alpha]} + \lambda\sum_{i=1}^{n}\frac{x_i e^{-\lambda\alpha x_i}}{1-e^{-\lambda\alpha x_i}} = 0 \quad (7)$$

and

$$\frac{\partial\, log\ \ell}{\partial\lambda} = \frac{2n}{\lambda} - \frac{n(1+\lambda)}{\lambda(1+\alpha)+2+\alpha} + \alpha\sum_{i=1}^{n}\frac{x_i e^{-\lambda\alpha x_i}}{1-e^{-\lambda\alpha x_i}} - \sum_{i=1}^{n} x_i = 0 \quad (8)$$

3. Moments Estimators

It is observed by Asgharzadehet al. that if X follows new weighted Lindley distribution, then

$$E(X)=\frac{(1+\alpha)^3(\lambda+2)-(1+\alpha)\lambda-2}{\lambda(1+\alpha)[\lambda\alpha(1+\alpha)+\alpha(2+\alpha)]}$$

$$E(X^2)=2\frac{(1+\alpha)^4(\lambda+3)-(1+\alpha)\lambda-3}{\lambda^2(1+\alpha)^2[\lambda(1+\alpha)+2+\alpha]^2}$$

The moments of estimators of the new weighted Lindley distribution can be obtained by equating the first two theoretical moments of (3) with the sample moments $\frac{1}{n}\sum_{i=1}^{n}x_i$ and $\frac{1}{n}\sum_{i=1}^{n}x_i^2$,

$$\frac{1}{n}\sum_{i=1}^{n}x_i=\frac{(1+\alpha)^3(\lambda+2)-(1+\alpha)\lambda-2}{\lambda(1+\alpha)[\lambda\alpha(1+\alpha)+\alpha(2+\alpha)]} \tag{9}$$

$$\frac{1}{n}\sum_{i=1}^{n}x_i^2=2\frac{(1+\alpha)^4(\lambda+3)-(1+\alpha)\lambda-3}{\lambda^2(1+\alpha)^2[\lambda(1+\alpha)+2+\alpha]^2}$$

(10)

4. Maximum product spacing estimates

The maximum product spacing (MPS) method has been proposed by Cheng and Amin which is based on an idea that the differences of the consecutive points should be identically distributed.
The geometric mean of the differences is given as

$$GM=\sqrt[n+1]{\prod_{i=1}^{n+1}D_i} \tag{11}$$

where, the difference D_i is defined as

$$D_i=\int_{x_{(i-1)}}^{x_{(i)}}f\,(x,\beta)dx;\;\; i=1,2,\dots,n+1. \tag{12}$$

where, $F(x_{(0)},\beta)=0$ and $F(x_{(n+1)},\beta)=1$. The MPS estimator $\hat{\beta}_{PS}$ of β is obtained by maximizing the geometric mean (GM) of the differences. Substituting (3) in (12) and taking logarithm of the above expression, we will have

$$LogGM=\frac{1}{n+1}\sum_{i=1}^{n+1}\log\left[F(x_{(i)},\beta)-F(x_{(i-1)},\beta)\right] \tag{13}$$

The MPS estimator $\hat{\beta}_{PS}$ of β can be obtained as the simultaneous solution of the following non-linear equation:

$$\frac{\partial LogGM}{\partial \beta} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F'_{\beta}(x_{(i)},\beta) - F'_{\beta}(x_{(i-1)},\beta)}{F(x_{(i)},\beta) - F(x_{(i-1)},\beta)}\right] = 0 \tag{14}$$

5. Methods of Minimum Distances

In this subsection we present three estimation methods for λ and σ based on the minimization, with respect to λ and σ, of the goodness-of-fit statistics. This class of statistics is based on the difference between the estimate of the cumulative distribution function and the empirical distribution function.

5.1 Method of Cramér-von-Mises

To motivate our choice of Cramer-von Mises type minimum distance estimators, MacDonald provided empirical evidence that the bias of the estimator is smaller than the other minimum distance estimators. Thus, The Cramé-von Mises estimates $\hat{\alpha}_{CME}$ and $\hat{\lambda}_{CME}$ of the parameters α and λ are obtained by minimizing, with respect to α and λ, the function:

$$C(\alpha,\lambda) = \frac{1}{12n} + \sum_{i=1}^{n}\left(F(x_{i:n}|\alpha,\lambda) - \frac{2i-1}{2n}\right)^2. \tag{15}$$

These estimates can also be obtained by solving the non-linear equations:

$$\sum_{i=1}^{n}\left(F(x_{i:n}|\alpha,\lambda) - \frac{2i-1}{2n}\right)\Delta_1(x_{i:n}|\alpha,\lambda) = 0,$$
$$\sum_{i=1}^{n}\left(F(x_{i:n}|\alpha,\lambda) - \frac{2i-1}{2n}\right)\Delta_2(x_{i:n}|\alpha,\lambda) = 0.$$

5.2 Methods of Anderson-Darling and Right-tail Anderson-Darling

The Anderson-Darling test was developed T.W. Anderson and D.A. Darling as an alternative to other statistical tests for detecting sample distributions departure from normality. The Anderson-Darling estimates $\hat{\alpha}_{ADE}$ and $\hat{\lambda}_{ADE}$ of the parameters α and λ are obtained by minimizing, with respect to α and λ, the function:

$$A(\alpha,\lambda) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\log F\,(x_{i:n}|\alpha,\lambda) + \log \overline{F}\,(x_{n+1-i:n}|\alpha,\lambda)\right\}. \tag{16}$$

These estimates can also be obtained by solving the non-linear equations:

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_1(x_{i:n}|\alpha,\lambda)}{F(x_{i:n}|\alpha,\lambda)} - \frac{\Delta_1(x_{n+1-i:n}|\alpha,\lambda)}{\overline{F}(x_{n+1-i:n}|\alpha,\lambda)}\right] = 0,$$

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_2(x_{i:n}|\alpha,\lambda)}{F(x_{i:n}|\alpha,\lambda)} - \frac{\Delta_2(x_{n+1-i:n}|\alpha,\lambda)}{\overline{F}(x_{n+1-i:n}|\alpha,\lambda)}\right] = 0.$$

The Right-tail Anderson-Darling estimates $\hat{\alpha}_{RTADE}$ and $\hat{\lambda}_{RTADE}$ of the parameters α and λ are obtained by minimizing, with respect to α and λ, the function:

$$R(\alpha,\lambda) = \frac{n}{2} - 2\sum_{i=1}^{n} F\,(x_{i:n}|\alpha,\lambda) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log \overline{F}\,(x_{n+1-i:n}|\alpha,\lambda). \tag{17}$$

6. Least square estimates

The least square estimators and weighted least square estimators were proposed by Swain, Venkataraman and Wilson to estimate the parameters of Beta distributions.

Let $x_{(1)}, x_{(2)}, \dots, x_{(n)}$ be the ordered sample of size n drawn the new weighted Lindley distribution population (3). Then, the expectation of the empirical cumulative distribution function is defined as

$$E\left[F\left(X_{(i)}\right)\right] = \frac{i}{n+1};\ i = 1,2,\dots,n \tag{18}$$

The least square estimates (LSEs) $\hat{\beta}_{LS}$ of β are obtained by minimizing

$$Z(\beta) = \sum_{i=1}^{n}\left(F\left(x_{(i)},\beta\right) - \frac{i}{n+1}\right)^2 \tag{19}$$

Therefore, $\hat{\beta}_{LS}$ of β can be obtained as the solution of the following equation:

$$\frac{\partial Z(\beta)}{\partial \beta} = \sum_{i=1}^{n} F'_{\beta}\,(x_{(i)},\beta)\left(F\left(x_{(i)},\beta\right) - \frac{i}{n+1}\right) = 0 \tag{20}$$

The weighted least square estimators of the unknown parameters can be obtained by minimizing

$$\sum_{j=1}^{n} w_j \left[F(X_{(j)}) - \frac{j}{n+1}\right]^2$$

with respect to β. The weights w_j are equal to $\frac{1}{V(X_{(j)})} = \frac{(n+1)^2(n+2)}{j(n-j+1)}$. Therefore, in this case, the weighted least square estimators of β, say $\hat{\beta}_{WLSE}$, can be obtained by minimizing

$$\sum_{j=1}^{n} \left[1 - \frac{1}{1+\beta}\left[\beta + (1+\beta x_j)e^{-\beta x_j}\right]e^{-\beta x_j} - \frac{j}{n+1}\right]^2$$

with respect to β.

7. Percentile Estimators

If the data come from a distribution function which has a closed form, then we can estimate the unknown parameters by fitting straight line to the theoretical points obtained from the distribution function and the sample percentile points. This method wasoriginally suggested by Kao (1958, 1959) and it has been used for Weibull distribution and for generalized exponential distribution. In this paper, we apply the same technique for the two-parameter EG distribution. Since,

$$F(x) = -\frac{e^{-\lambda x}\left\{(1+\alpha)^2(1+\lambda+\lambda x) - [\lambda(1+\alpha)(1+x)+1]e^{-\alpha\lambda x}\right\}}{\alpha\lambda(1+\lambda) + \alpha(2+\alpha)}$$

therefore

$$x_p = \frac{1}{\lambda}\left[\frac{(1+\alpha)^2(1+\lambda+\lambda x_p) - e^{-\alpha\lambda x_p}\left[\lambda(1+\alpha)(x_p+1)+1\right]}{\alpha(1-p)[\lambda(1+\alpha)+2+\alpha]}\right].$$

Let $X_{(j)}$ be the jth order statistic, i.e, $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$. If p_j denotes some estimate of $G(x_{(j)};\alpha,\lambda)$, then the estimate of α and λ can be obtained by minimizing

$$\sum_{j=1}^{n}\left(x_{(j)} - \frac{1}{\lambda}\left[\frac{(1+\alpha)^2(1+\lambda+\lambda x_p) - e^{-\alpha\lambda x_p}\left[\lambda(1+\alpha)(x_p+1)+1\right]}{\alpha(1-p)[\lambda(1+\alpha)+2+\alpha]}\right]\right)^2$$

with respect to α and λ.

We call the corresponding estimates as the percentile estimators or PCE's.

Several estimators of p_j can be used here, see for example Mann, Schafer and Singpurwalla. In this paper, we consider $p_j = \frac{j}{n+1}$.

8. L-Moments Estimators

In this section we provide the L-moments estimators, which can be obtained as the linear combinations of order statistics. The L-moments estimators were originally proposed by Hosking (1990), and it is observed that the L-moments estimators are more robust than the usual moment estimators. The L-moment estimators are also obtained along the same way as the ordinary moment estimators, i.e. by equating the sample L-moments with the population L-moments. L-moment estimation provides an alternative method of estimation analogous to conventional moments and have the advantage that they exist whenever the mean of the distribution exists, even though some higher moments may not exist, and are relatively robust to the effects of outliers (Hosking, 1994). Hosking(1990) states that the L-moment estimators are reasonably efficient when compared to the maximum likelihood estimators for distributions such as the normal distribution, the Gumbel distribution, and the GEV distribution.
In this case the L-moments estimators can be obtained by equating the first two sample L-moments with the corresponding population L-moments. The first two sample L-moments are

$$l_1 = \frac{1}{n}\sum_{i=1}^{n} x_{(i)}, \qquad l_2 = \frac{2}{n(n-1)}\sum_{i=1}^{n}(i-1)x_{(i)} - l_1$$

and the first two population L-moments are

$$\lambda_1 = E(X_{1:1}) = E(X)$$

$$\lambda_2 = \frac{1}{2}[E(X_{2:2}) - E(X_{2:1})],$$

The L-moments estimators $\hat{\alpha}_{LME}$ and $\hat{\lambda}_{LME}$ of the parameters α and λ can be obtained by solving numerically the following equations:

$$\lambda_1(\hat{\alpha}_{LME}, \hat{\lambda}_{LME}) = l_1 \qquad ,$$
$$\lambda_2(\hat{\alpha}_{LME}, \hat{\lambda}_{LME}) = l_2.$$

9. Simulation algorithms and study

To generate a random sample of size n from New weighted Lindleydistribution, we follow the following steps:

1. Set n, $\Theta = (\beta)$ and initial value x^0.
2. Generate $U \sim Unifrom(0,1)$.
3. Update x^0 by using the Newton's formula

$$x^\star = x^0 - R(x^0, \Theta)$$

where, $R(x^0, \Theta) = \frac{F_X(x^0,\Theta)-U}{f_X(x^0,\Theta)}$, $F_X(.)$ and $f_X(.)$ are cdf and pdf of new weighted Lindley distribution, respectively.

1. If $|x^0 - x^\star| \le \epsilon$, (very small, $\epsilon > 0$ tolerance limit), then store $x = x^\star$ as a sample from New weighted Lindleydistribution.
2. If $|x^0 - x^\star| > \epsilon$, then, set $x^0 = x^\star$ and go to step 3.
3. Repeat steps 3-5, n times for $x_1, x_2, \ldots, x_n$ respectively.

10. Comparison study of the proposed estimators

This subsection deals with the comparisons study of the proposed estimators in terms of their mean square error on the basis of simulates sample from pdf or new weighted Lindleydistribution with varying sample sizes. For this purpose, we take $\beta = 3$, λ =2 arbitrarily and $n = 10,20,\ldots,50$. All the algorithms are coded in R, a statistical computing environment and we used algorithm given above for simulations purpose.

We calculate Maximum likelihood estimation(MLE),methods of moments(MM), modified method of moments(MME), Least square estimatior(LSE), Percentile estimation(PE),L-moments estimation(LME),Maximum product spacing(MPS),Cramer-von Mises estimation(CME) and Anderson Darling estimation(ADE) of β based on each generated sample. This proses is repeated 1000 of times, and average estimates and corresponding mean square errors are computed and also reported in Table 1 and 2.

From Table 1 and Table 2 it can be observed that as sample size increases the mean square error decreases, it proves the consistency of the estimators. The least square estimator(LSE) of parameters$\beta$and $\lambda$is superior than the others methods of estimation.

Table 1. Estimates and mean square errors (in IInd row of each cell) of the proposed estimators with varying sample size for β

n	MLE	ME	MME	LSE	PCE	LME	MPS	CME	ADE
10	2.8442	2.38901	2.6470	3.1442	2.8901	3.6470	3.3442	2.6901	2.2470
	1.1717	1.4215	0.6798	1.0717	1.3215	1.0798	1.2717	1.3215	0.9798
20	2.9193	2.9866	2.8221	3.0893	2.9866	3.2240	3.2293	2.7866	2.5240
	0.3925	0.5933	0.2969	0.32145	0.4933	0.5969	0.3925	0.4933	0.7969
30	2.94573	2.9886	2.8583	3.0173	2.9886	3.2283	3.1273	2.8886	2.7583
	0.2298	0.2882	0.1902	0.2098	0.2832	0.3902	0.2598	0.3182	0.3942

40	2.97235	2.9952	2.8805	3.0115	2.9952	3.2005	3.1115	2.9052	2.7805
	0.1590	0.2119	0.1345	0.1420	0.2119	0.2385	0.1590	0.3119	0.2385
50	3.0965	2.9934	2.8932	3.0110	2.9934	3.0932	3.1065	2.9134	2.8032
	0.1252	0.1682	0.1141	0.0552	0.1682	0.2141	0.1252	0.1982	0.2141

Table 2. Estimates and mean square errors (in IInd row of each cell) of the proposed estimators with varying sample size for λ

n	MLE	ME	MME	LSE	PCE	LME	MPS	CME	ADE
10	1.7554	1.2790	4.6570	1.1554	1.7901	1.6570	4.2554	1.6901	1.4570
	1.1717	1.5415	0.6797	1.0717	1.2415	1.0797	1.4717	1.2415	0.9797
40	1.9192	1.9766	4.7441	1.0792	1.9766	4.4450	4.4492	1.7766	1.5450
	0.2945	0.5922	0.4969	0.24155	0.5922	0.5969	0.2945	0.5922	0.7969
20	1.95572	1.9776	4.7572	1.0172	1.9776	4.4472	4.1472	1.7776	1.7572
	0.4497	0.4774	0.1904	0.4097	0.4724	0.2904	0.4597	0.2174	0.2954
50	1.97425	1.9954	4.7705	1.0115	1.9954	4.4005	4.1115	1.9054	1.7705
	0.1590	0.4119	0.1255	0.1540	0.4119	0.4275	0.1590	0.2119	0.4275
50	1.0965	1.9925	4.7924	1.0110	1.9925	4.0924	4.1065	1.9125	1.7024
	0.1454	0.1674	0.1151	0.0554	0.1674	0.4151	0.1454	0.1974	0.4151

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