ON RATIONAL TRIGONOMETRIC SPLINE

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Abstract. In the introduction part of this paper are given some general notations, definitions and examples for rational interpolation and interpolation by splines. Next is achieved and proved assessment and comparison, in form of inequalities, between the surfaces that are formed by the graphs formed of the second derivative of spline and the second derivative of function interpolated by that spline, in a given segment, considering boundary conditions. Finally for a constructed smooth rational cubic trigonometric spline with shape parameters, are studied its properties and manipulated with parameters to constrain the shape of interpolant region and illustrated with numerical examples.

Key words: *Rational Interpolation, Cubic Spline, Trigonometric spline, Smooth Function*

Introduction

Definition 1. Let $\Delta : a = x_0 < x_1 < x_2 < ... < x_n = b$ be a subdivision of the segment a,b . A function $S^m_{f\Delta} :\ a,b\ \to R, m \in N$ is called a spline of degree *m* with respect to the subdivision Δ, if $S^m_{f\Delta}$ is $m-1$ $-$times continuously differentiable on a,b and if the restriction of $S^m_{f\Delta}$ to each subinterval $x_{i-1}, x_i\ , i = 1,2,...,n-1,n$ reduces to a polynomial of degree smaller or equal to *m*. By S^m_n we denote all splines of degree *m* for a fixed subdivision of segment in *n* pieces. S^m_n is linear space of dimension *m*+*n*.

Definition 2. Let $\Delta : a = x_0 < x_1 < x_2 < ... < x_{k+1} = b$ be a subdivision of the segment a,b . The space of trigonometric splines is defined as $S^M_{T_m\Delta} =\ s : s|_{x_i, x_{i+1}} \in T_m, i = 0,1,2,...,k \wedge D^{j-1}s\ \ x_i^-\ = D^{j-1}s\ \ x_i^+$,

$j = 1,2,...,m - m_i, i = 1,2,...,k$, such that $M = (m_1,...,m_k)$ be a vector which elements satisfy $1 \le m_i \le m$, $D = d/dx$ is differential and operator $T_m = \text{span}\{\phi_{m-i-1}(x)\varphi_i(x)\}_{i=0}^{m-1}$ be the space of trigonometric polynomials of order m, where $\phi_k(x) = \phi(kx), \varphi_k(x) = \varphi(kx), k \in N$ $\phi(x) = \sin \alpha x, \varphi(x) = \sin \alpha x$ and $\text{span}(A) = \left\{ \sum_{i=1}^{k} \lambda_i v_i \,/\, v_i \in A \wedge \lambda_i \in R \right\}$.

Here we present some examples of splines:

1. $B^0 -$splines: these functions are splines of order 1 and $B^1 -$splines: these functions are splines of order 1, and reaches a peak at $x = x_{i+1}$ and is upward (downward) sloping for $x < x_{i+1}$ $(x > x_{i+1})$.

$$B_i^0(x) = \begin{cases} 1, & x_i \le x \le x_{i+1} \\ 0, & \text{else} \end{cases} \quad \text{and} \quad B_i^1(x) = \begin{cases} \dfrac{x - x_i}{x_{i+1} - x_i}, & x_i \le x \le x_{i+1} \\ \dfrac{x_{i+2} - x}{x_{i+2} - x_{i+1}}, & x_{i+1} \le x \le x_{i+2} \\ 0, & \text{else} \end{cases}.$$

2. Higher order spline functions are defined by the recursion:

$$B_i^n(x) = \left(\frac{x - x_i}{x_{i+n} - x_i} \right) B_i^{n-1}(x) + \left(\frac{x_{i+n+1} - x}{x_{i+n+1} - x_{i+1}} \right) B_{i+1}^{n-1}(x)$$

3. Cubic splines: these functions are splines of order 4 and its analytic expression is as $S_{f\Delta} = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$ for $x \in [x_i, x_{i+1}]$ where

$$a_i = f(x_i) = f_i,\ b_i = \frac{f_{i+1} - f_i}{h_{i+1}} - \frac{h_{i+1}}{6}(2M_i + M_{i+1}),\ c_i = \frac{M_i}{2},\quad d_i = \frac{M_{i+1} - M_i}{6h_{i+1}}$$

$M_i = \left(S_{f_i\Delta}\right)''$ are solutions of system of linear equations

$$\mu_i M_{i-1} + 2M_i + v_i M_{i+1} = \lambda_{i,} \quad i = \overline{1, n-1}$$

as amended by the given boundary conditions,

$$h_i = x_i - x_{i-1}, \quad \mu_i = \frac{h_i}{h_i + h_{i+1}}, \quad v_i = 1 - \mu_i, \quad \lambda_i = 6f\left(x_{i-1}, x_i, x_{i+1}\right).$$

4. Cubic trigonometric spline: is presented as $\mathrm{span}\left(1, x, \sin x, \cos x, \sin 2x, \cos 2x\right)$. These splines have many similar properties to cubic B-splines and their corresponding spline curves and surfaces can interpolate directly some control points without solving system of equations or inserting some additional control points. The curves can be used to represent exactly straight line segment, circular arc, elliptic arc, parabola and some transcendental curves such as circular helix.

Some assessments of spline and fuction interpolated by it

Theorem 1. Let $f\left(x\right) \in C^2\left[0,1\right]$ be periodic function. If $S_{f\Delta}$ be cubic spline interpolation of function $f\left(x\right)$ than holds the inequality

$\max\limits_{1 \le i \le n-1} \left| \left(S_{f_i\Delta}\right)'' \right| \le 6 \max\limits_{1 \le i \le n-1} \left| f\left(x_{i-1}, x_i, x_{i+1}\right) \right|$, where

$\Delta = \left\{0 = x_0 < x_1 < \ldots < x_n = 1\right\}$.

Proof. $M_i = \left(S_{f_i\Delta}\right)''$ are solutions of system of linear equations

$\mu_i M_{i-1} + 2M_i + v_i M_{i+1} = \lambda_{i,} \quad i = \overline{1, n-1}$ where

$h_i = x_i - x_{i-1}, \quad v_i = \frac{h_{i+1}}{h_i + h_{i+1}}$, $\mu_i + v_i = 1$ and $\lambda_i = 6f\left(x_{i-1}, x_i, x_{i+1}\right)$. Let

$\max\limits_{1 \le i \le n-1} \left|M_i\right| = \left|M_k\right|, k \in \left[1, n-1\right]$ and since $\mu_i, v_i \in \left(0,1\right), \forall i \in \overline{1, n-1}$

than the relation

$$\max_{1\le i\le n}|\lambda_i| \ge |\lambda_k| \ge 2|M_k| - \mu_k|M_{k-1}| - \nu_k|M_{k+1}| \ge |M_k|(2-\mu_k-\nu_k) =$$
$$=|M_k| = \max_{1\le i\le n-1}|M_i|.$$

Is valid.

Theorem 2 . Let $f(x)\in C^p[a,b]$ and the spline $s\in S_m^n, m=2p-1, p\in N, p\ge 2$ be its interpolant, respectively $s(x_i)=f(x_i), i=\overline{1,n}$, and satisfies the boundary conditions $s^{(i)}(a)=f^{(i)}(a), s^{(i)}(b)=f^{(i)}(b), i=\overline{1,p-1}$

than holds $\int_a^b \left[f^{(p)}(x)-s^{(p)}(x)\right]^2 dx = \int_a^b \left(\left[f^{(p)}(x)\right]^2 - \left[s^{(p)}(x)\right]^2\right) dx.$

Proof. We have that $\int_a^b \left[f^{(p)}(x)-s^{(p)}(x)\right]^2 dx = \int_a^b \left(\left[f^{(p)}(x)\right]^2 - \left[s^{(p)}(x)\right]^2\right) dx - 2R$ where $R=\int_a^b \left[f^{(p)}(x)-s^{(p)}(x)\right]s^{(p)}(x)\,dx$. Since $f(x)\in C^p[a,b]$ and $s\in C^{m-1}[a,b]$ has continuous derivatives of order m, by $p-1$ repeated integrations and using the boundary conditions and since $s^{(m+1)}(x)=0$ is obtained that

$$R=(-1)^{p-1}\int_a^b \left[f'(x)-s'(x)\right]s^{(m)}(x)\,dx =$$
$$=(-1)^{p-1}\sum_{i=1}^n \int_{x_{i-1}}^{x_i} \left[f'(x)-s'(x)\right]s^{(m)}(x)\,dx =$$
$$=(-1)^{p-1}\left(\sum_{i=1}^n \left[f(x)-s(x)\right]s^{(m)}(x)\,dx\Big|_{x_{i-1}}^{x_i}\right)=0.$$

Corollary 1. Let $f(x)=0$ and the spline $s\in S_m^n, m=2p-1,\ p\in N, p\geq 2$ be its interpolant, respectively $s(x_i)=f(x_i),\ i=\overline{1,n}$, and satisfies the boundary conditions $s^{(i)}(a)=f^{(i)}(a), s^{(i)}(b)=f^{(i)}(b),\ i=\overline{1,p-1}$ than $s=0$.

Proof. For $f(x)=0$, from theorem 1, we have:

$$\int_a^b\left(0-s^{(p)}(x)\right)^2dx=0-\int_a^b\left(s^{(p)}(x)\right)^2dx\Rightarrow\int_a^b\left(s^{(p)}(x)\right)^2dx=0\Rightarrow s^{(p)}(x)=0.$$

Now from boundary conditions since $s^{(i)}(a)=0, i=\overline{1,p-1}\Rightarrow s=0.$

Theorem 3. Let $f(x)\in C^2[a,b]$ be a function and $S_{f\Delta}$ be cubic spline that interpolates the function $f(x)$ at the nodes $\Delta=\{a=x_0<x_1<...<x_n=b\}$ and satisfies the boundary conditions $S'_{f\Delta}(a)=f'(a),\ S'_{f\Delta}(b)=f'(b)$ than $\int_a^b\left[S''_{f\Delta}(x)\right]dx\leq\int_a^b\left[f''(x)\right]dx$ is valid.

Proof. Let us form the difference $D(x)=f(x)-S_{f\Delta}(x)$ and now we have $\int_a^b\left[f''(x)\right]^2dx=\int_a^b\left[S''_{f\Delta}(x)\right]^2dx+\int_a^b\left[D''(x)\right]^2dx+2\int_a^b S''_{f\Delta}(x)D''(x)dx$. By integration by parts of last integral we have $\int_a^b S''_{f\Delta}(x)D''(x)dx=S''_{f\Delta}(x)D'(x)\Big|_a^b-\int_a^b S'''_{f\Delta}(x)D'(x)dx$. First term of the right hand of the equation is zero because of boundary conditions and now The integral in the second term can be divided into subintervals, as follows:

$$-\int_a^b S'''_{f\Delta}(x)D'(x)dx=-\sum_{i=1}^{n-1}\int_{x_i}^{x_{i+1}}S'''_{f\Delta}(x)D'(x)dx,$$

$$\int_{x_i}^{x_{i+1}}S'''_{f\Delta}(x)D'(x)dx=S'''_{f\Delta}(x)D(x)\Big|_{x_i}^{x_{i+1}}-\int_{x_i}^{x_{i+1}}S^{(4)}_{f\Delta}(x)D(x)dx=0-0=0.$$

So

$$\int_a^b \left[f''(x)\right]^2 dx - \int_a^b \left[D''(x)\right]^2 dx = \int_a^b \left[S''_{f\Delta}(x)\right]^2 dx \Rightarrow \int_a^b \left[S''_{f\Delta}(x)\right] dx \leq \int_a^b \left[f''(x)\right] dx.$$

Rational cubic trigonometric spline

Let $(t_i, f_i) \,/\, i = \overline{0, n+1}$ be a given set of data points, such that $a = t_0 < t_1 < \ldots < t_n < t_{n+1} = b$. The C[1]-continous, piecewise rational cubic function based on the function values is defined as $P(t) = \dfrac{p_i(t)}{q_i(t)}$, $i = \overline{0, n-1}$, where

$$p_i(t) = (1-\sin\varphi)^3 \alpha_i f_i + \sin\varphi(1-\sin\varphi)(3-\sin\varphi) u_i +$$
$$+ \cos\varphi(1-\cos\varphi)(3-\cos\varphi) v_i + (1-\cos\varphi)^3 \beta_i f_{i+1}$$

and

$$q_i(t) = (1-\sin\varphi)^3 \alpha_i + \sin\varphi(1-\sin\varphi)(3-\sin\varphi) + \cos\varphi(1-\cos\varphi)(3-\cos\varphi) +$$
$$+ (1-\cos\varphi)^3 \beta_i$$

Such that $h_i = t_{i+1} - t_i$, $\varphi(t) = \dfrac{\pi(t-t_i)}{2h_i}$, $t \in [t_i, t_{i+1}]$,

$u_i = f_i + \dfrac{2h_i\alpha_i}{3\pi}(f_{i+1} - f_i)$, $v_i = f_{i+1} - \dfrac{2h_i\beta_i}{3\pi}\dfrac{f_{i+2} - f_{i+1}}{h_{i+1}}$ and $\alpha_i, \beta_i > 0$. It is obvious that this rational cubic trigonometric function satisfies the relation $P(t) > 0$, $P(t_i) = f_i$ and $P'(t_i) = \dfrac{f_{i+1} - f_i}{h_i}$. With $g(t)$ we denote the broken line or linear piece-wise curve defined on $[t_0, t_n]$ with joining points $(t_i, f_i) \,/\, i = \overline{0, n+1}$ with node system $t_0 < t_1 < \ldots < t_n < t_{n+1}$ such that $f(t_i) \geq (\leq) g(t_i)$ and $P(t) \geq (\leq) g(t)$ where $i = \overline{0, n+1}$ and $t \in [t_0, t_{n+1}]$.

We suppose that $P(t) \geq g(t)$, $\dfrac{p_i(t)}{q_i(t)} \geq g(t)$,

$p_i(t) - q_i(t)g(t) = M_i(t) \geq 0$, respectively

$$M_i(t) = \{(1-\sin\varphi)^3 \alpha_i f_i + \sin\varphi(1-\sin\varphi)(3-\sin\varphi)u_i +$$
$$+\cos\varphi(1-\cos\varphi)(3-\cos\varphi)v_i + (1-\cos\varphi)^3 \beta_i f_{i+1}\} - \{(1-\sin\varphi)^3 \alpha_i +$$
$$+\sin\varphi(1-\sin\varphi)(3-\sin\varphi) + \cos\varphi(1-\cos\varphi)(3-\cos\varphi) +$$
$$+(1-\cos\varphi)^3 \beta_i\}\left\{\left(1-\frac{2\varphi}{\pi}\right)g_i + \frac{2\varphi}{\pi} g_{i+1}\right\} \geq 0$$

where $g_i = g(t_i)$ and $g_{i+1} = g(t_{i+1})$. since

$$M_i(t) = (1-\sin\varphi)^3 \alpha_i A_{i,1} + \sin\varphi(1-\sin\varphi)(3-\sin\varphi)A_{i,2} +$$
$$+\cos\varphi(1-\cos\varphi)(3-\cos\varphi)A_{i,3} + (1-\cos\varphi)^3 \beta_i A_{i,4} \geq 0$$

$$A_{i,1} = \left(1-\frac{2\varphi}{\pi}\right)(f_i - g_i) + \frac{2\varphi}{\pi}(f_{i+1} - g_{i+1}) - \frac{2\varphi}{\pi}(f_{i+1} - f_i),$$
$$A_{i,2} = A_{i,1} + \frac{2\alpha_i}{3\pi}(f_{i+1} - f_i),$$
$$A_{i,3} = A_{i,4} - \frac{2h_i\beta_i}{3\pi}\frac{(f_{i+2} - f_{i+1})}{h_{i+1}}, \quad A_{i,4} = A_{i,1} + (f_{i+1} - f_i).$$

According to the last relation the last sufficient condition for the curve $P(t)$ to lie above the straight line $g(t)$ on segment $[t_i, t_{i+1}]$ is stated in this theorem:

Theorem 4. For the given set $\{(t_i, f_i, g_i) \; / \; i = \overline{0, n+1}\}$ such that $f(t_i) \geq g(t_i)$, $i = \overline{0, n+1}$ the sufficient condition for the curve $P(t)$ to lie above the straight line $g(t)$ on segment $[t_i, t_{i+1}]$ is that for the positive parameters α_i and β_i must be valid the relation $A_{i,k} \geq 0, \quad k = 1,2,3,4$. For equidistant nodes the relation $A_{i,3}$ takes the form $A_{i,3} = A_{i,4} - \dfrac{2\beta_i}{3\pi}(f_{i+2} - f_{i+1})$, and the boundness of the region where the interpolation curve is situated, can be stated this result:

Theorem 5. For the given set $\{t_i, f_i, g_i, g_i^* \ / \ i=\overline{0,n+1}\}$ such that $g(t_i) \le f(t_i) \le g^*(t_i)$, $i=\overline{0,n+1}$. $f(t_i) \ge g(t_i)$, $i=\overline{0,n+1}$ the sufficient condition for the rational cubic trigonometric curve $P(t)$ to lie above the straight line $g(t)$ and under the straight line $g^*(t)$ on segment $[t_i, t_{i+1}]$ is that for the positive parameters α_i and β_i must be valid the relation $A_{i,k} \ge 0, \quad k=1,2,3,4$ defined as above and $B_{i,k} \le 0, \quad k=1,2,3,4$ where

$$B_{i,1} = \left(1-\frac{2\varphi}{\pi}\right)(f_i - g^*_i) + \frac{2\varphi}{\pi}(f_{i+1} - g^*_{i+1}) - \frac{2\varphi}{\pi}(f_{i+1} - f_i),$$

$$B_{i,2} = B_{i,1} + \frac{2\alpha_i}{3\pi}(f_{i+1} - f_i),$$

$$B_{i,3} = B_{i,4} - \frac{2\beta_i}{3\pi}(f_{i+2} - f_{i+1}), \quad B_{i,4} = B_{i,1} + f_{i+1} - f_i.$$

Example: the interpolation data and positive parameters α_i dhe $\beta_i, \quad i=1,2,3$ are given in the table below, and fulfilling the conditions from theorem 4 and 5 in order the curve $P(t)$ to lie $g^*(t)$ and $g(t)$, which can be written as:

$$g^*(t) = \begin{cases} -0,16t + 0.107, & t \in [0, 0.5] \\ 0,12t - 0.033, & t \in [0.5, 1] \\ -0,08t + 0.167, & t \in [1, 1.5] \\ 0,04t - 0,013, & t \in [1.5, 2] \end{cases}$$

and

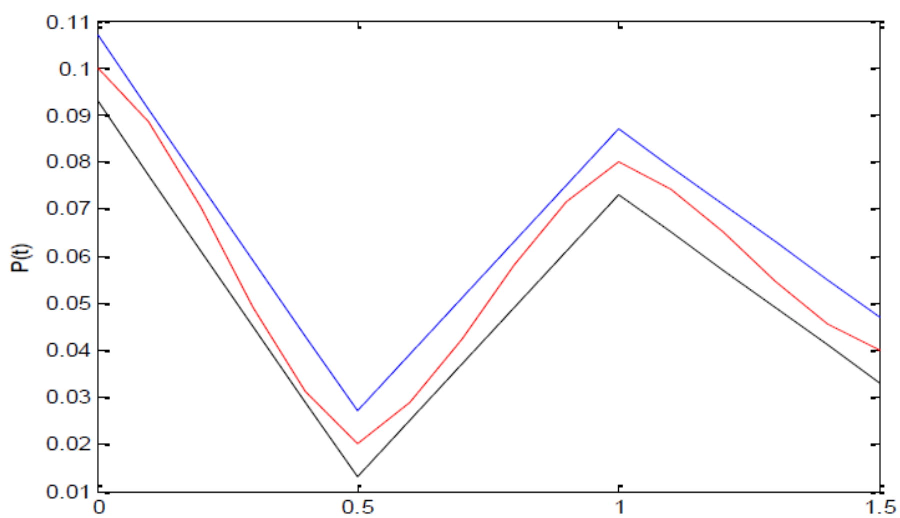
$$g(t) = \begin{cases} -0,16t + 0.093, & t \in [0, 0.5] \\ 0,12t - 0.047, & t \in [0.5, 1] \\ -0,08t + 0.153, & t \in [1, 1.5] \\ 0,04t - 0,027, & t \in [1.5, 2] \end{cases}.$$

t_i	i	$g^*(t_i)$	$f(t_i)$	$g(t_i)$
0	1	0,107	0,1	0,093
0,5	2	0,027	0,02	0,013
1	3	0,087	0,08	0,073
1,5	4	0,047	0,04	0,033
2	5	0,067	0,06	0,053

and

i	α_i	β_i
1	0,001123	0,0011423
2	0,001555	0,00124
3	0,001510	0,0011905

These data are represented in the graph below where the graphs of curves $g^*(t)$, $P(t)$ and $g(t)$ are in blue, red and black color respectively.



Conclusion

Some assessments and between the surfaces that are formed by the graphs formed of the second derivative of spline and the second derivative of function interpolated by that spline, in a given segment, considering boundary conditions are given. Examples of splines including rational trigonometric cubic spline discussing the conditions to fulfill of shape parameters for constraining spline curve to be bounded between two parallel straight lines.

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