APPLICATION TO TRUNCATED CAUCHY POWER ODD FRÉCHET-EXPONENTIAL DISTRIBUTION

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Abstract

In this paper, we describe different methods of estimation for the modified Exponential distribution called the Truncated Cauchy Power Odd Fréchet-Exponential distribution. These methods are, maximum likelihood estimators, least squares estimation, weighted least squares, maximum product spacing estimates, method of Cramer-von-Misses, Anderson–Darling, right-tail Anderson–Darling methods and compare them using extensive numerical simulations. Applications reveals that the model proposed can be very useful in fitting real data. Two applications are carried out on real data to show the potentiality of the proposed family.

Key words: Exponential Distribution, Estimation Methods, Kolmogorov-Smirnov Test.

Përmbledhje

Në këtë artikull, do të përshkruajmë metoda të ndryshme vlerësimi për shpërndarjen e modifikuar Eksponenciale e emërtuar shpërndarja e Copëtuar Koshi në Fuqi Tek të Fréchet-Ekspoenciale. Metodat e vlerësimit janë metoda e përgjasisë maksimale, metoda e katrorëve më të vegjël, sipas peshave, metoda e prodhimit të distancave maksimale, metoda Kramer-von-Majsis, metoda Anderson Darling dhe Anderson Darling nga e djathta. Rezultatet e fituara nga metodat e vlerësimit do të krahasohen midis tyre duke përdorur simulime numerike. Zbatimet tregojnë që shpërndarja e modifikuar është shumë e përdorur në të dhënat reale. Në material janë ndërtuar dy zbatime me të dhëna reale të cilat tregojnë përparësinë e shpërndarjes së propozuar.

Fjalë kyçe: Shpërndarja eksponenciale, metodat e vlerësimit, testi Kollmogorov-Smirnov.

Introduction

Many experts have offered a wide range of methods for including an extra parameter in distributions, and all these new families have been used to describe data from a wide variety of fields, covering engineering, economics, biological studies, environmental sciences, and so on. These families are created by adding an additional shape parameter to the parent distribution to enhance the capabilities and validity of the data modeling.

The exponential distribution is the most applied statistical distribution for problems in reliability, as an important subject of statistics, such as engineering, biostatistics, and other industrial areas.

Many authors have been proposed extended or modified of exponential distribution. Ahuja *et al.* (1967) made the generalized exponential distribution. Balakrishnan (1985) introduced order statistics of half logistic exponential distribution. Nadarajah *et al.* (2006) introduced some properties of beta exponential distribution. Ristić *et al* (2015) proposed the Marshall-Olkin generalized exponential distribution. Aldahlan *et al* (2020) applied some estimation methods and applications to the odd exponentiated half-logistic exponential distribution. Also, Arbër *et al* (2021) introduced an extended exponential distribution which is generated from the Generalized Odd Half-Logistic Family.

In this paper, we describe different methods of estimation for the Truncated Cauchy Power Odd Fréchet-Exponential distribution which is generated from the Truncated Cauchy Power Odd Fréchet-G family of distributions. These methods include maximum likelihood, least squares estimation, weighted least squares estimation, Cramer–von Mises, maximum product of spacings, Anderson–Darling and right-tail Anderson–Darling methods. Numerical simulation experiments are conducted to assess the performance of the so obtained estimators developed from these methods. The Kolmogorov-Smirnov test is used as the criterion for comparison. Also, we made two applications on real data to show the potentiality of the proposed family.

Definition 1.1

A continuous random variable X is said to have an Exponential distribution if it has probability density function, Harry (2003):

$$g(x;\theta) = \begin{cases} \theta e^{-\theta x} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{1}$$

where, $\theta > 0$ is called the rate of distribution.

The cumulative density function of exponential distribution is given by (2):

$$G(x;\theta) = \begin{cases} 1 - e^{-\theta x} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{2}$$

Shrahili *et al*. (2021), proposed a new family of distributions based on the odd Fréchet-G and truncated Cauchy power-G families.

Definition 1.2 For any continuous baseline cumulative distribution function (cdf) $G(x)$ the cumulative distribution function of the Truncated Cauchy Power Odd Fréchet generator is given by (3):

$$F(x;\alpha,\lambda,\xi) = \frac{4}{\pi}\arctan e^{-\lambda\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^{\alpha}}, \; x > 0 \tag{3}$$

The probability density function (pdf) associated with equation (3) is given by

$$f(x;\alpha,\lambda,\xi) = \frac{4\lambda\alpha g(x,\xi)\left(1-G(x,\xi)\right)^{\alpha-1} e^{-\lambda\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^{\alpha}}}{\pi G(x,\xi)^{\alpha+1}\left(1+e^{-2\lambda\left(\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^{\alpha}\right)^{\alpha}}\right)} \tag{4}$$

By putting equations (1) and (2) as a baseline function to (3) and (4), we get the Truncated Cauchy Power Odd Fréchet-Exponential distribution.

Definition 1.3 A random variable X is said to have the Truncated Cauchy Power Odd Fréchet-Exponential distribution if its probability density function is defined as:

$$f(x;\alpha,\lambda,\theta) = \frac{4\lambda\alpha\theta e^{-\theta x}\left(e^{-\theta x}\right)^{\alpha-1} e^{-\lambda\left(\frac{e^{-\theta x}}{1-e^{-\theta x}}\right)^{\alpha}}}{\pi\left(1-e^{-\theta x}\right)^{\alpha+1}\left(1+e^{-2\lambda\left(\left(\frac{e^{-\theta x}}{1-e^{-\theta x}}\right)^{\alpha}\right)^{\alpha}}\right)}, \; x > 0, \alpha > 0, \theta > 0, \lambda > 0 \tag{5}$$

and cumulative distribution function associated with equation (5) is given by:

$$F(x;\alpha,\lambda,\theta) = \frac{4}{\pi}\arctan e^{-\lambda\left(\frac{e^{-\theta x}}{1-e^{-\theta x}}\right)^{\alpha}}, \; x > 0, \alpha > 0, \theta > 0, \lambda > 0. \tag{6}$$

Figure 1 and 2 illustrates some of the possible shapes of the probability density function and cumulative density function, of the Truncated Cauchy Power Odd Fréchet-Exponential distribution for selected values of the parameters α, θ and λ respectively.

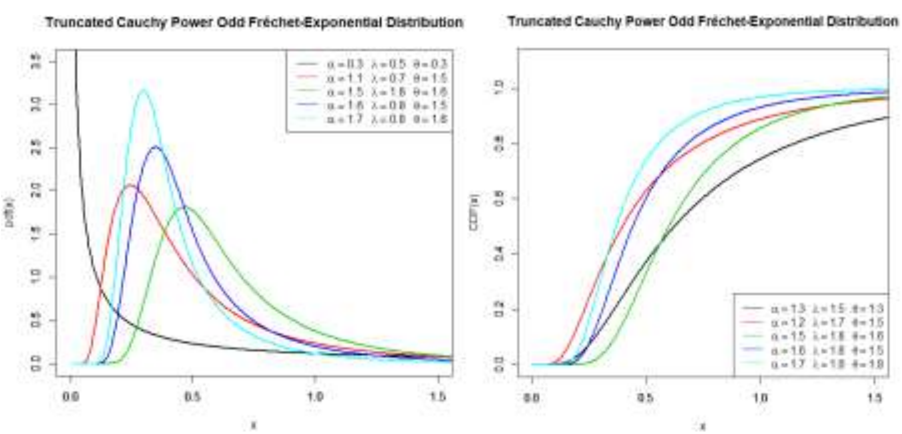

Figure 1. The PDF and CDF of the Truncated Cauchy Power Odd Fréchet-Exponential distribution for selected values of the parameters α, θ and λ.

2. Reliability Analysis

2.1 Survival Function

The survivor function simply indicates the probability that the event of interest has not yet acquired by time t. Thus, if T denotes time until death, $S(t)$ denotes probability of surviving beyond time t.

The survival function of the Truncated Cauchy Power Odd Fréchet-Exponential distribution is given:

$$S(t) = P(T \geq t) = \int_t^{\infty} f(x)\,dx = 1 - F(t) = 1 - \frac{4}{\pi}\arctan e^{-\lambda\left(\frac{e^{-\theta t}}{1-e^{-\theta t}}\right)^{\alpha}} \tag{7}$$

The hazard rate function (failure rate) of a lifetime random variable X with Truncated Cauchy Power Odd Fréchet-Exponential distribution is given by:

$$h(t;\alpha,\lambda,\theta) = \frac{4\lambda\alpha\theta\, e^{-\theta t}\left(e^{-\theta t}\right)^{\alpha-1} e^{-\lambda\left(\frac{e^{-\theta t}}{1-e^{-\theta t}}\right)^{\alpha}}}{\pi\left(1-e^{-\theta t}\right)^{\alpha+1}\left(1+e^{-2\lambda\left(\left(\frac{e^{-\theta t}}{1-e^{-\theta t}}\right)^{\alpha}\right)^{\alpha}}\right)\left(1-\frac{4\arctan\left(e^{-\lambda\left(\frac{e^{-\theta t}}{1-e^{-\theta t}}\right)^{\alpha}}\right)}{\pi}\right)} \tag{8}$$

The cumulative hazard rate function of a lifetime random variable X with Truncated Cauchy Power Odd Fréchet-Exponential distribution is given by:

$$H(t;\alpha,\lambda,\theta) = \int_0^t h(u)du = -ln\left(1-\frac{4}{\pi}\arctan e^{-\lambda\left(\frac{e^{-\theta t}}{1-e^{-\theta t}}\right)^{\alpha}}\right) \qquad (9)$$

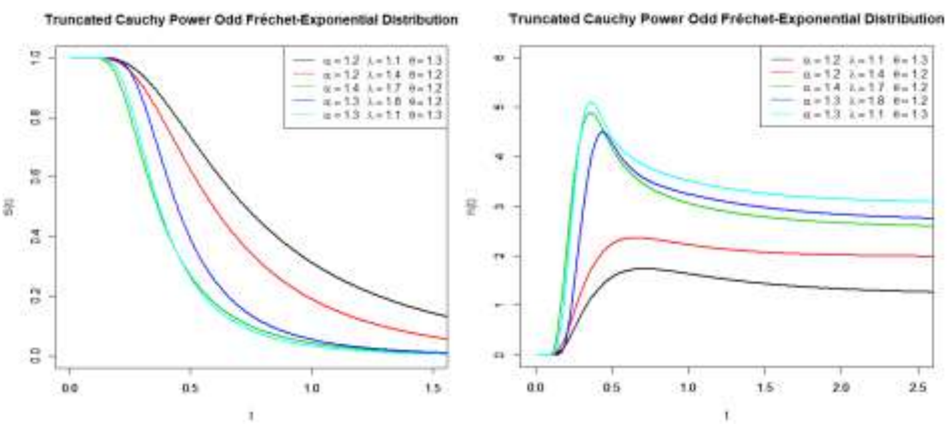

Figure 3. The survivor function and hazard rate function of the Truncated Cauchy Power Odd Fréchet-Exponential distribution for selected values of the parameters α, θ and λ.

3. Order Statistics

Order statistics are among the most fundamental tools in non-parametric statistics and inference. For $X_1, X_2, \cdots, X_n$, i.i.d. random variables X_k is the kth smallest X, usually called the kth order statistic.

$X_{(1)}$ is therefore the smallest X and $X_{(1)} = min(X_1, X_2, \cdots, X_n)$. Similarly, $X_{(n)}$ is the largest X and $X_{(n)} = max(X_1, X_2, \dots, X_n)$

For $X_1, X_2, \cdots, X_n$ i.i.d. continuous random variables with pdf (5) and cdf (6) the density of the minimum is:

$$f_{(1)}(x) = nf(x)[1-F(x)]^{n-1} = n\left(\frac{4\lambda\alpha\theta e^{-\theta x}(e^{-\theta x})^{\alpha-1} e^{-\lambda\left(\frac{e^{-\theta x}}{1-e^{-\theta x}}\right)^{\alpha}}}{\pi(1-e^{-\theta x})^{\alpha+1}\left(1+e^{-2\lambda\left(\left(\frac{e^{-\theta x}}{1-e^{-\theta x}}\right)^{\alpha}\right)^{\alpha}}\right)}\right)\times\left(1-\frac{4}{\pi}\arctan e^{-\lambda\left(\frac{e^{-\theta x}}{1-e^{-\theta x}}\right)^{\alpha}}\right)^{n-1} \qquad (10)$$

For $X_1, X_2, \cdots, X_n$ i.d.d. continuous random variables pdf (5) and cdf (6) the density of the maximum is:

$$f_{(n)}(x) = nf(x)\,[F(x)]^{n-1} = n\left(\frac{4\lambda\alpha\theta\, e^{-\theta x}\left(e^{-\theta x}\right)^{\alpha-1} e^{-\lambda\left(\frac{e^{-\theta x}}{1-e^{-\theta x}}\right)^{\alpha}}}{\pi\left(1-e^{-\theta x}\right)^{\alpha+1}\left(1+e^{-2\lambda\left(\left(\frac{e^{-\theta x}}{1-e^{-\theta x}}\right)^{\alpha}\right)}\right)}\right) \times \left(\frac{4}{\pi}\arctan e^{-\lambda\left(\frac{e^{-\theta x}}{1-e^{-\theta x}}\right)^{\alpha}}\right)^{n-1}$$

(11)

For $X_1, X_2, \cdots, X_n$ iid continuous random variables with pdf (5) and cdf (6) the density of the kth order statistic is:

$$f_{(k)}(x) = n\binom{n-1}{k-1} f(x)[F(x)]^{k-1}[1-F(x)]^{n-k} =$$

$$n\binom{n-1}{k-1}\left(\frac{4\lambda\alpha\theta\, e^{-\theta x}\left(e^{-\theta x}\right)^{\alpha-1} e^{-\lambda\left(\frac{e^{-\theta x}}{1-e^{-\theta x}}\right)^{\alpha}}}{\pi\left(1-e^{-\theta x}\right)^{\alpha+1}\left(1+e^{-2\lambda\left(\left(\frac{e^{-\theta x}}{1-e^{-\theta x}}\right)^{\alpha}\right)}\right)}\right)\left(\frac{4}{\pi}\arctan e^{-\lambda\left(\frac{e^{-\theta x}}{1-e^{-\theta x}}\right)^{\alpha}}\right)^{k-1} \times \left(1-\frac{4}{\pi}\arctan e^{-\lambda\left(\frac{e^{-\theta x}}{1-e^{-\theta x}}\right)^{\alpha}}\right)^{n-k}$$

(12)

4. Inference

In this section, we estimate the parameters of the Truncated Cauchy Power Odd Fréchet-Exponential distribution by following the method of maximum likelihood estimators, least squares estimation, weighted least squares, maximum product spacing estimates, method of Cramer-von-Misses, Anderson–Darling, right-tail Anderson–Darling methods and compare them using extensive numerical simulations.

4.1 Maximum Likelihood estimators

Let $x_1, x_2, \cdots, x_n$ be a random sample of size n from the Truncated Cauchy Power Odd Fréchet-Exponential distribution. The log-likelihood function for the parameters α, θ and λ can be expressed as:

$$\ell(\alpha,\theta,\lambda) = n\log\left(\frac{4\lambda}{\pi}\right) + n\log(\alpha) + \sum_{i=1}^{n}\log\left(\theta e^{-\theta x_i}\right) + (\alpha-1)\sum_{i=1}^{n}\log\left(e^{-\theta x_i}\right) - -(\alpha+1)\sum_{i=1}^{n}\log\left(1-e^{-\theta x_i}\right) - \lambda\sum_{i=1}^{n}\left(\frac{e^{-\theta x_i}}{1-e^{-\theta x_i}}\right)^{\alpha} - \sum_{i=1}^{n}\log\left(1+e^{-2\lambda\left(\frac{e^{-\theta x_i}}{1-e^{-\theta x_i}}\right)^{\alpha}}\right)$$

(13)

Thus, the MLE of the parameters α, θ and λ can be obtained by setting the derivates of equations (13) to zero and solving them iteratively, using numerical methods such as the Newton-Raphson iteration method. Alternatively, the log-likelihood in equation (13) can be directly maximized, using any standard non-linear optimization tool.

4.2 Least Squares estimation

The least squares estimators and weighted least squares estimators (LSEs) were proposed by Swain, Venkatraman and Wilson (1988) to estimate the parameters of a Beta distribution. The LSEs of the unknown parameters of Truncated Cauchy Power Odd Fréchet-Exponential distribution can be obtained by minimizing:

$$Z(\alpha,\lambda,\theta)=\sum_{j=1}^{n}\left(F\left(x_{(j)}|\alpha,\lambda,\theta\right)-\frac{j}{n+1}\right)^{2} \tag{14}$$

with respect to the unknown parameters α,λ and θ. $F(\cdot)$ denotes the distribution of the Truncated Cauchy Power Odd Fréchet-Exponential distribution and $E\left(F\left(x_{(j)}\right)\right)=\frac{j}{n+1}$ is the expectation of the empirical cumulative distribution function.

Therefore, $\tilde{\alpha}_{LSE}, \tilde{\lambda}_{LSE}$ and $\tilde{\theta}_{LSE}$ of α,λ,θ can be obtained as the solution of the

following system of equations:

$$\begin{aligned}
\frac{\partial LS(\alpha,\lambda,\theta)}{\partial\alpha}&=2\sum_{j=1}^{n}F'_{\alpha}\left(x_{(j)}|\alpha,\lambda,\theta\right)\left(F\left(x_{(j)}|\alpha,\lambda,\theta\right)-\frac{j}{n+1}\right)=0\\
\frac{\partial LS(\alpha,\lambda,\theta)}{\partial\lambda}&=2\sum_{j=1}^{n}F'_{\lambda}\left(x_{(j)}|\alpha,\lambda,\theta\right)\left(F\left(x_{(j)}|\alpha,\lambda,\theta\right)-\frac{j}{n+1}\right)=0\\
\frac{\partial LS(\alpha,\lambda,\theta)}{\partial\theta}&=2\sum_{j=1}^{n}F'_{\theta}\left(x_{(j)}|\alpha,\lambda,\theta\right)\left(F\left(x_{(j)}|\alpha,\lambda,\theta\right)-\frac{j}{n+1}\right)=0
\end{aligned} \tag{15}$$

4.3 Weighted Least Squares

The weighted least squares estimators (WLSEs) of the unknown parameters can be obtained by minimizing

$$\sum_{j=1}^{n}\omega_j\left(F\left(x_{(j)}\right)-\frac{j}{n+1}\right)^{2} \tag{16}$$

with respect to α,λ and θ, where ω_j denotes the weight function at the jth

point, which is equal to

$$\omega_j=\frac{1}{V(F(X_{(j)})}\frac{(n+1)^2(n+2)}{j(n-j+1)}$$

The weighted least square estimates (WLSEs) $\tilde{\alpha}_{WLS}, \tilde{\lambda}_{WLS}, \tilde{\theta}_{WLS}$ of α,λ,θ can

be obtained by minimizing

$$WLS(\alpha,\lambda,\theta) = \sum_{j=1}^{n} \frac{(n+1)^2(n+2)}{j(n-j+1)} \left(\frac{4}{\pi} \arctan e^{-\lambda\left(\frac{e^{-\theta x_j}}{1-e^{-\theta x_j}}\right)^{\alpha}} - \frac{j}{n+1} \right)^2 \quad (17)$$

Therefore, the estimators $\tilde{\alpha}_{WLSE}, \tilde{\lambda}_{WLSE}, \tilde{\theta}_{WLSE}$ can be obtained from the first partial derivative of the (17) with respects to α, λ and θ set the result equal to zero.

4.4 Maximum Product Spacing Estimates

The maximum product spacing (MPS) method has been proposed by Cheng and Amin (1983). This method is based on an idea that the differences (Spacings) of the consecutive points should be identically distributed. The geometric mean of the differences is given as

$$GM = \sqrt[n+1]{\prod_{i=1}^{n=1} D_i}$$

where, the difference D_i is defined as

$$D_i = \int_{x(i-1)}^{x(i)} f(x|\alpha,\lambda,\theta)\,dx,\ i = 1,2,\dots,n+1. \quad (18)$$

Also, $F\left(x_{(0)},\alpha,\lambda,\theta\right) = 0$ and $F\left(x_{(n+1)},\alpha,\lambda,\theta\right) = 1$.

The MPS estimators $\tilde{\alpha}_{ML}, \tilde{\lambda}_{ML}, \tilde{\theta}_{ML}$ of α, λ, θ is obtained by maximizing the geometric mean (GM) of the differences. Substituting (4) in (18) and taking logarithm of the above expression, we will have

$$\log GM = \sum_{i=1}^{n+1} \log\left[F\left(x_{(i)},\alpha,\lambda,\theta\right) - F\left(x_{(i-1)},\alpha,\lambda,\theta\right)\right] \quad (19)$$

The MPS estimators $\tilde{\alpha}_{ML}, \tilde{\lambda}_{ML}, \tilde{\theta}_{ML}$ of α, λ, θ can be obtained as the simultaneous solution of the following non-linear equations:

$$\frac{\partial logGM}{\partial \alpha} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F'_{\alpha}(x_{(i)}|\alpha,\lambda,\theta)-F'_{\alpha}(x_{(i-1)}|\alpha,\lambda,\theta)}{F(x_{(i)}|\alpha,\lambda,\theta)-F(x_{(i-1)}|\alpha,\lambda,\theta)}\right] = 0$$

$$\frac{\partial logGM}{\partial \lambda} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F'_{\lambda}(x_{(i)}|\alpha,\lambda,\theta)-F'_{\beta}(x_{(i-1)}|\alpha,\lambda,\theta)}{F(x_{(i)}|\alpha,\lambda,\theta)-F(x_{(i-1)}|\alpha,\lambda,\theta)}\right] = 0 \quad (20)$$

$$\frac{\partial logGM}{\partial \theta} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F'_{\theta}(x_{(i)}|\alpha,\lambda,\theta)-F'_{\theta}(x_{(i-1)}|\alpha,\lambda,\theta)}{F(x_{(i)}|\alpha,\lambda,\theta)-F(x_{(i-1)}|\alpha,\lambda,\theta)}\right] = 0$$

4.5 Method of Cramer-von-Misses

The Cramér-von-Mises, Barrie D. at el. (1991), estimator (CME) is a type of minimum distance estimators, which is based on the Cramér-von-Mises statistic motivates the choice of Cramér-von-Mises type minimum distance estimators providing empirical evidence that the bias of the estimator is smaller than the other minimum distance estimators. The Cramér-von-Mises estimates $\tilde{\alpha}_{CME}, \tilde{\lambda}_{CME}, \tilde{\theta}_{CME}$ of parameters $\alpha, \lambda, \theta,$ of Truncated Cauchy Power Odd Fréchet-Exponential distribution are obtained by minimizing, with respect to α, θ, λ the function:

$$C(\alpha,\lambda,\theta) = \frac{1}{12n} + \sum_{i=1}^{n}\left(F\left(x_{(i)}|\alpha,\lambda,\theta\right) - \frac{2i-1}{n}\right)^2 \quad (21)$$

These estimates can be obtained by solving the nonlinear equations:

$$\frac{\partial C(\alpha,\lambda,\theta)}{\partial \alpha} = \sum_{i=1}^{n} 2\left(F\left(x_{(i)}|\alpha,\lambda,\theta\right) - \frac{2i-1}{n}\right)\frac{\partial F(x_{(i)}|\alpha,\lambda,\theta)}{\partial \alpha} = 0$$

$$\frac{\partial C(\alpha,\lambda,\theta)}{\partial \lambda} = \sum_{i=1}^{n} 2\left(F\left(x_{(i)}|\alpha,\lambda,\theta\right) - \frac{2i-1}{n}\right)\frac{\partial F(x_{(i)}|\alpha,\lambda,\theta)}{\partial \lambda} = 0 \quad (22)$$

$$\frac{\partial C(\alpha,\lambda,\theta)}{\partial \theta} = \sum_{i=1}^{n} 2\left(F\left(x_{(i)}|\alpha,\lambda,\theta\right) - \frac{2i-1}{n}\right)\frac{\partial F(x_{(i)}|\alpha,\lambda,\theta)}{\partial \theta} = 0$$

4.6 Anderson–Darling and Right-tail Anderson–Darling methods

Another type of minimum distance estimators is based on Anderson-Darling statistic (T. W. Anderson and D. A. Darling (1952), (1954)) and is known as the Anderson-Darling estimator (ADE).

The Anderson-Darling estimates of the parameters are obtained by minimizing, with respect to α, λ and θ, the function:

$$A(\alpha,\lambda,\theta) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[logF\left(x_{(i)}|\alpha,\lambda,\theta\right) + \log\bar{F}\left(x_{(n+1-i)}|\alpha,\lambda,\theta\right)\right] \quad (23)$$

so,

$$A(\alpha,\lambda,\theta) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[log\left(\frac{4}{\pi}\arctan e^{-\lambda\left(\frac{e^{-\theta x_i}}{1-e^{-\theta x_i}}\right)^{\alpha}}\right) + log\left(1-\frac{4}{\pi}\arctan e^{-\lambda\left(\frac{e^{-\theta x_{(n+1-i)}}}{1-e^{-\theta x_{(n+1-i)}}}\right)^{\alpha}}\right)\right] \tag{24}$$

The Right-tail Anderson-Darling (RADE) estimates (Ye, Y. at el (2017)) of the parameters are obtained by minimizing, with respect to α, λ and θ, the function:

$$R(\alpha,\lambda,\theta) = \frac{n}{2} - 2\sum_{i=1}^{n} F\left(x_{(i)}\middle|\alpha,\lambda,\theta\right) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log \bar{F}\left(x_{(n+1-i)}|\alpha,\lambda,\theta\right) \tag{25}$$

so,

$$R(\alpha,\lambda,\theta) = \frac{n}{2} - 2\sum_{i=1}^{n}\left(\frac{4}{\pi}\arctan e^{-\lambda\left(\frac{e^{-\theta x_i}}{1-e^{-\theta x_i}}\right)^{\alpha}}\right) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)log\left(1-\frac{4}{\pi}\arctan e^{-\lambda\left(\frac{e^{-\theta x_{(n+1-i)}}}{1-e^{-\theta x_{(n+1-i)}}}\right)^{\alpha}}\right) \tag{26}$$

5. Simulation Study

In this section, we perform a Monte Carlo simulation study to evaluate the performance of different estimation methods for estimating the parameters of the Truncated Cauchy Power Odd Fréchet-Exponential distribution. We compare the proposed estimators by using the Kolmogorov-Smirnov test. This procedure is based on the K-S statistic:

$$D_n = \max_x |F_n(x) - F(x|\alpha,\lambda,\theta)|$$

where $\max_x$ denotes the maximum of the set of distances, $F_n(x)$ is the empirical distribution function, and $F(x|\ \alpha,\lambda,\theta)$ is the cumulative distribution function.

Firstly, we provided an algorithm to generate a random sample from the Truncated Cauchy Power Odd Fréchet-Exponential distribution for given values of its parameters and sample size n. The following procedure was adopted:

1. Set n, $\Theta = (\alpha,\lambda,\theta)$ and initial value x^0.
2. Generate $U \sim Uniform(0,1)$.
3. Update x^0 by using the Newton's formula

$x^* = x^0 - R(x^0, \Theta)$

where, $R(x_0,\Theta) = \frac{F_X(x^0,\Theta)-U}{f_X(x^0,\Theta)}$, $F_X(\cdot)$ and $f_X(\cdot)$ are cdf and pdf of the Truncated Cauchy Power Odd Fréchet-Exponential distribution, respectively.

4. If $|x^0 - x^*| \leq \epsilon$ (very small, $\epsilon > 0$ tolerance limit), then store $x = x^*$ as a sample from Truncated Cauchy Power Odd Fréchet-Exponential distribution.
5. If $|x^0 - x^*| > \epsilon$, then set $x^0 = x^*$ and go to step 3.
6. Repeat steps 3-5, n times for $x_1, x_2, \ldots, x_n$ respectively.

For this purpose, we take $\alpha = 1, \lambda = 1, \theta = 0.5$ arbitrarily and $n = 10, 20, \ldots, 50$. All the algorithms are coded in R, a statistical computing environment and we used algorithm given above for simulations purpose.

From the results of the simulation study, it is observed that the method of **Maximum Likelihood Estimation** (MLE) is the most efficient method compared to others for estimating the parameters of the Truncated Cauchy Power Odd Fréchet-Exponential distribution since it has the minimum value of Kolmogorov-Smirnov test (Table 1).

In addition, the MLE estimators have good theoretical properties (Cheng & Amin, 1983) such as consistency, asymptotic efficiency, normality, and invariance, so we conclude that the MLE estimators should be used for estimating the parameters of the Truncated Cauchy Power Odd Fréchet-Exponential distribution.

Table 1: The methods of estimation and its respective Kolmogorov-Smirnov test value.

	Methods of Estimations	**Kolmogorov-Smirnov**	**Ranking**
1	**Maximum Likelihood Estimation** (MLE)	**0.040142**	**1**
2	Least Square Estimation (LSE)	0.043068	2
3	Weighted Least Square Estimation (WLSE)	0.051816	5
4	Maximum Product Spacings Estimation (MPSE)	0.060706	7
5	Cramer Von Mises Estimation (CVME)	0.043714	3
6	Anderson-Darling Estimation ((ADE)	0.049249	4
7	Right-tail Anderson-Darling Estimation (RADE)	0.054777	6

The empirical and the fitted cumulative distribution function of the Truncated Cauchy Power Odd Fréchet-Exponential distribution are presented in Figure 4 for the simulated data.

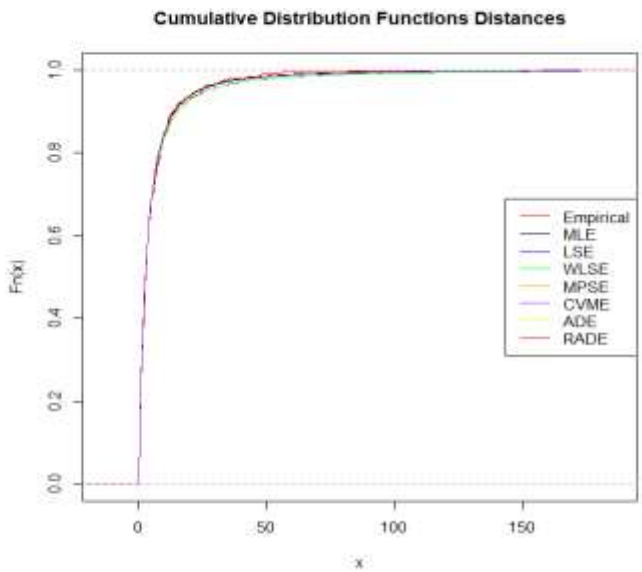

Figure 4. Empirical and fitted CDF of the Truncated Cauchy Power Odd Fréchet-Exponential distribution for the simulated data.

Applications

In this section, we provide two applications for complete data sets to model the Truncated Cauchy Power Odd Fréchet-Exponential distribution. These datasets are associated with the engineering area. The first dataset relates to the study of failure times of 84 windshields for a particular model of aircraft (the unit for measurement is 1000 hours) that was first discussed by Ramos et al (2013). The second dataset relates to the study of service times of 63 aircraft windshields (the unit for measurement is 1000 hours) that was discussed by Tahir et al (2015). We compare the Truncated Cauchy Power Odd Fréchet-Exponential (TCPOF-E) distribution to Exponentiated Exponential (Ahuja and Nash, 1967), Nadarajah-Highlight Exponential (Nadarajah and Haghighi, 2011), Half Logistic Exponential (Balakrishnan, 1985), Exponential (Nadarajah and Kotz, 2006) distributions.

Table 2. Parameters estimates and standard errors along with goodness-of-fit for the failure time of 84 windshield data set.

Model	**ML Estimates**	**Standard Error**	**Log-Likelihood**	**AIC**	**BIC**
TCPOF-Exp	Alpha=0.8321 Lambda=0.3658	0.284 0.018	120.128	246.254	252.573

	Theta=0.8547	0.012			
E-Exp	Alpha=0.6739 Theta=0.5965	0.086 0.036	121.470	249.204	255.228
NH-Exp	Alpa=0.3562 Theta=0.6532	0.235 0.512	121.603	250.937	255.586
HL-Exp	Theta=0.6779	0.0942	126.490	251.003	253.634
Exp	Theta=0.6598	0.0754	128.325	262.074	258.401

Table 3. Parameters estimates and standard errors along with goodness-of-fit for the failure time of 63 aircraft windshields data set.

Model	**ML Estimates**	**Standard Error**	**Log-Likelihood**	**AIC**	**BIC**
TCPOF-Exp	Alpha=0.9566 Lambda=0.6872 Theta=0.9761	0.2131 0.5124 0.3211	117.204	230.485	221.236
E-Exp	Alpha=0.2512 Theta=0.3527	0.8452 0.3984	117.475	231.211	222.758
NH-Exp	Alpha=0.6251 Theta=0.3652	0.2845 0.6248	118.235	231.265	223.001
HL-Exp	Theta=0.3251	0.1241	122.365	241.352	224.652
Exp	Theta=0.6852	0.0893	129.125	245.236	227.623

In these applications, the model parameters are estimated by the method of maximum likelihood. The Akaike information criterion (AIC), Bayesian information criterion (BIC) and log-likelihood are computed to compare the fitted models. In general, the smaller the values of these statistics, the better the fit to the data. From results above table 2 and table 3 we get that the TCPOF-E has the minimum values, so this model is the optimal model for fitting the data.

Conclusions

In this paper we described different methods of estimation for the Exponential distribution called the Truncated Cauchy Power Odd Fréchet-Exponential distribution. These methods are, namely maximum likelihood estimators, least squares estimation, weighted least squares, maximum product spacing estimates, method of Cramer-von-Misses, Anderson–Darling, right-tail Anderson–Darling The results from these methods are compared them using extensive numerical simulations. Eventually, real-world data applications are utilized to demonstrate the practicality of TCPOF-E distribution. In future, we may use this model to study the statistical inference of it using Bayesian estimation under different censored schemes.

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