AN ALBANIAN EXAMPLE OF THE LEONTIEF DYNAMIC MODEL: AN APPROACH THROUGH MATRIX ALGEBRA AND DIFFERENTIAL EQUATIONS

RIGENA SEMA[1], BESIANA ÇOBANI[2], XHILDA MERKAJ (DHAMO)[3]

[1]Department of Mathematics, Faculty of Natural Science, Tirana University

[2]Department of Mathematics, Faculty of Natural Science, Tirana University

[3]Department of Applied Mathematics, Faculty of Natural Science, Tirana University

e-mail: rigena.sema@fshn.edu.al

Abstract

Leontief's dynamic model is used to study the growth of gross national product and national income. According to this model, from the mathematical side, production activities are represented by means of the matrix of input-output coefficients and the capital coefficient matrix. The solutions obtained are not always positive. In economics, the focus is to obtain positive solutions, so efforts have been made by various researchers to treat the positive solutions of these models by means of algebraic methods and differential equation. In this paper, the economic development of Albania is studied according to three different sectors, which are agriculture, manufacturing, and services, for a period of 10 years, relying on INSTAT data.

Key words: Dynamic Leontief model, Drazin inverse matrix, Leontief matrix, dynamic system, difference equation.

Përmbledhje

Modeli dinamik i Leontief-it përdoret për të studjuar rritjen e produktit kombëtar bruto dhe të ardhurave kombëtare. Sipas këtij modeli, nga ana matematikore, aktivitetet prodhuese paraqiten me anë të matricës së koeficientëve input-output dhe matricës *capital coefficient*. Zgjidhjet që përftohen jo gjithmonë janë pozitive. Në ekonomi interesi është për të përftuar zgjidhje pozitive, ndaj nga kërkues të ndryshëm janë bërë përpjekje që të trajtohen zgjidhjet pozitive të këtyre modeleve me anë të metodave algjebrike si dhe ekuacioneve diferenciale. Në këtë punim studiohet zhvillimi ekonomik i Shqipërisë sipas tre sektoreve të ndryshem të cilët janë agrikultura, manifaktura dhe shërbimet për një periudhë 10 vjeçare, duke u mbështetur në të dhënat e INSTAT. Ky studim bëhet nëpërmjet aparatit teorik të algjebrës lineare dhe ekuacioneve diferenciale. Këto modele, më pas testohen në Matlab dhe EXCEL për të parë cila nga metodat paraqet rezultatin më të mirë.

Fjalë kyçe: Modeli dinamik i Leontief, matrica inverse e Drazin-it, matrica e Leontef-it, sistem dinamik, ekuacione me diferenca.

Introduction

The dynamic input-output models consider the changes in the sectors depending on the time variable t, differently from the common input-output model that assume the economic structure is constant, which means does not take in consideration the time variable.

The discrete difference equation that describes the dynamic Leontief model (Leontief, 1970)

$$X(t) = AX(t) + B[X(t+1) - X(t)] + Y(t) \quad (1)$$

provides a recursive method frequently used in practical applications. $X(t)$ represents the non-negative r-dimensional vector of gross industrial production in year t and $Y(t)$ represents the r-dimensional vector of final consumption demands for commodities in year t. The amount of goods required to produce one unit of output is indicated by the nxn matrix of input coefficients A; and the nxn matrix of capital coefficients B, where the element bij represent the capital stock of good *i* required to produce one unit of output in sector *j*. Since these coefficients are constant, they also serve as a representation of the additional input and stock requirements for the growth of one unit of output.

Kendrek (1972), Bergendorff (1973), Livesey (1973), Lungerberger and Arbel (1977) have addressed the question of the location of the Leontief trajectory model (1) when B is not invertible. In their work the positive nature of the solution is not taken into consideration. But Szyld and Moledo *et.al.* (1988) suggested an algebraic-geometric approach, to find a positive solution in case of closed Leontief dynamic input-output model.

Jódar and Merello (2010, b). First the homogeneous solution of system (1) is presented, then using Drazin inverse matrix they give sufficient conditions for the initial vector such that model (1) has a positive solution.

The construction of power series solutions of dynamic Leontief input-output models with a potential one-matrix function was studied by Jódar and Merello (2010, a). They assume that the data function is analytic around the origin.

In this paper we study the algebraic model and the analytic method in the case of the Albanian economy. The data were provided by INSTAT for the period 2009-2018.

2. Methodology

a) Algebraic method

A dynamic Leontief model of a multisector economy has the following structure:

$$X(t) = AX(t) + B[X(t+1) - X(t)] + Y(t) \quad (1)$$

where B is the capital coefficient matrix, A is the Leontief input-output matrix, $X(t)$ is the vector of output levels and $Y_{Y(t)}$ is the vector of final demands. It is assumed that A and B are known and time invariant. If vector $X(t) \in \mathbb{C}^r$ and system (1) operate over the period $t = 0,1,\cdots,N-1$, the set of equations represent a set of rN equation with $r(N+1)$ unknowns. A Leontief system trajectory is the name given to a solution of this set.

The trajectory of model (1) is

$$X(t+1) = B^{-1}[(I-A+B)X(t) - Y(t)] \qquad (2)$$

where the matrix B is invertible and the initial condition X_0 is given.

Usually, matrix B is not invertible since not all industries produce substantial capital items. The rank of matrix B may be smaller than the number of sectors.

The Model (1)'s is equivalent with the following

$$BX(t+1) = (I-A+B)X(t) - Y(t),\ 0 \leq t \leq N-1 \qquad (3)$$

The elements of matrix A satisfy the following conditions

$0 \leq a_{ij} \leq 1,\ \ 1 \leq i,j \leq r$

$\sum_{i=1}^{r} a_{ij} \leq 1,\ \ 1 \leq j \leq r$

$\sum_{i=1}^{r} a_{ij_0} < 1$ for some j_0 such that $1 \leq j_0 \leq r$,

because A is Leontief matrix. Then matrix $I-A$ is invertible and $(I-A)^{-1} > 0$.

For $\lambda = 1$ we have

$\lambda B - (B + I - A) = (\lambda - 1)B + A - I = A - I$

so, the system $BX(t+1) = (I-A+B)X(t)$ is tractable by S.L. Campbell, C.D. Meyer (1979). By using the following notation:

$$\hat{B} = (A-I)^{-1}B = -(I-A)^{-1}B \qquad (4)$$

$$(I - \widehat{A+B}) = -(I-A)^{-1}(I-A+B) = -I - (I-A)^{-1}B = \hat{B} - I \qquad (5)$$

and if $k = ind(\hat{B})$, where k is the least nonnegative integer such that null spaces of matrices $\hat{B}^k$ and $\hat{B}^{k+1}$ are equal, $N(\hat{B}^k) = N(\hat{B}^{k+1})$, and

$$\hat{\omega} = (I - \hat{B}\widehat{B^D}) \sum_{i=0}^{k-1} \left(\hat{B}(\hat{B}-I)^D\right)^i (\hat{B}-I)^D (B-I)^{-1}$$

we have the existence of a vector $q \in \mathbb{R}^r$ as a consistent initial vector of model (1) if and only if q lies in $\{\hat{\omega} + R(\widehat{B^k})\}$ and the general solution of (1) for $t \geq 1$ is given by

$$X(t) = \left(\hat{B}^D(\hat{B}-I)\right)^t \hat{B}\hat{B}^D q - \hat{B}^D \sum_{j=0}^{t-1} \left(\hat{B}^D(\hat{B}-I)\right)^{t-i-j} (I-A)^{-1} Y_j +$$

$$+(I-\hat{B}\hat{B}^{D})\sum_{i=0}^{k-1}\left(\hat{B}(\hat{B}-I)^{D}\right)^{i}(\hat{B}-I)^{D}(I-A)^{-1}Y_{t+i},\ t\geq 1. \qquad (6)$$

The next step is the identification of the criteria on the problem's data set such that $X(t)$ is nonnegative for $t=1,2,\cdots,N$. Jódar. L, Merello. P (2010, b) demonstrated that if $B\geq 0$, then $\hat{B}\leq 0$ and $(I-\widehat{A+B})\leq 0$ from ((4) and ((5)). They assumed that the diagonal elements of $\hat{B}$ are nonzero, different from one, and that the diagonal elements of $\hat{B}^{D}$ are nonzero and $\hat{B}^{D}\leq 0$. They defined

$$\alpha_n=\frac{\|I-\hat{B}\hat{B}^{D}\|\sum_{i=0}^{k-1}\|\hat{B}(\hat{B}-I)^{D}\|^{i}\|(\hat{B}-I)^{D}\|\|\widetilde{Y_{n+1}}\|}{\left(d_{max}(\hat{B}^{D})\right)^{n+1}\left(d_{max}(\hat{B}-I)\right)^{n}d_{max}(\hat{B})}+\frac{\sum_{i=0}^{n-1}\left[(\hat{B}^{D})_{min}\right]^{n-1}[(\hat{B}-I)_{min}]^{t-i-1}r^{t-i+1}\tilde{Y}_{i}}{\left(d_{max}(\hat{B}^{D})\right)^{n+1}\left(d_{max}(\hat{B}-I)\right)^{n}d_{max}(\hat{B})} \quad (7)$$

where

$$\alpha=max\{\alpha_i: 0\leq i\leq N\} \qquad (8)$$

From the above they proved the following theorem which demonstrates when a Leontief trajectory is nonnegative.

Theorem 2.1 Under the hypotheses there exists $q\in\mathbb{R}^{r}$ having all its components $q_i\geq\alpha$ where α is defined by (7) -(8) and the corresponding Leontief trajectory $X(t)$ given by (6) is nonnegative for $0\leq i\leq N$.

We emphasize that A^{D} is pseudo-inverses or the Drazin inverse of a singular matrix A, for more information Campbell. & Meyer, *et.al.*, (1976) or Gon´zales *et.al.*, (2002).

b) Analytic Method

Multisector economics are studied with the dynamic Leontief models which is given previously as model (1) are also represented as differential equation as follows

$$X(t)=A(t)X(t)+Y(t)+B(t)X'(t) \qquad (8)$$

where $A(t)$ is the Leontief input -output matrix function, $B(t)$ is the capital matrix function, $Y(t)$ is the demand vector function.

We assume that $A(t)$, $B(t)$, $Y(t)$ are analytic at $t=0$. Let $A(t)$ be a $\mathbb{C}^{r\times r}$ valued analytic function of the real variable t, with power series expansion at t=0,

$$A(t)=\sum_{m\geq 0}A_m t^{m}, \qquad |t|<r_0$$

If there exist $M>0$ such that $\|A(t)\|\leq M$ for $|t|=r<r_0$ then

$$\|A_m\|\leq\frac{M}{r^{n}} \qquad m\geq 0$$

We continue with the function $Y(t)$,

$$Y(t) = \sum_{m \geq 0} Y_m t^m \qquad |t| < r_0$$

having $\|Y_m\| \leq \frac{M}{t^m} \qquad m \geq 0.$

Considering equation (8) with the assumption of analicity of the input-output matrix function $A(t)$, the demand vector $Y(t)$ and the capital coefficient matrix function $B(t)$ in a neighborhood $|t| < r_0$ of the point t=0. We suppose that

$$A(t) = \sum_{m \geq 0} A_m t^m, B(t) = \sum_{m \geq 0} B_m t^m, Y(t) = \sum_{m \geq 0} Y_m t^m \qquad |t| < r_0, \qquad r_0 > 0$$

where A_m, B_m are matrices in $\mathbb{R}^{n \times n}$ and Y_m is a vector in $\mathbb{R}^n$ for $m \geq 0$. We rewrite equation (8) in the following form

$$B(t)X'^{(t)} + (A(t) - I)X(t) + Y(t) = 0$$

$$B(t)X'^{(t)} + \breve{A}(t)X(t) + Y(t) = 0 \qquad (9)$$

where

$\breve{A}(t) = A(t) - I = \sum_{m \geq 0} \breve{A}_m(t) t^m$ and $\breve{A}_0(0) = A_0 - I \; \breve{A}_m = A_m \; m \geq 1.$

We look for a solution $X(t)$ of (2) as

$$X(t) = \sum_{m \geq 0} X_m t^m \qquad (10)$$

where X_m are undetermined. Assuming the convernce of the series $\sum_{m \geq 0} X_m t^m$ we have

$$\breve{A}(t)X(t) = (\sum_{m \geq 0} \breve{A}_m t^m)(\sum_{m \geq 0} X_m t^m) = \sum_{m \geq 0} M_m t^m \qquad (11)$$

$$B(t)X'^{(t)} = (\sum_{m \geq 0} B_m t^m)(\sum_{m \geq 0}(m+1)X_{m+1} t^m) = \sum_{m \geq 0} C_m t^m \qquad (12)$$

where $M_m = \sum_{k=0}^{m} \breve{A}_{m-k} X_k$, $C_m = \sum_{k=0}^{m} (k+1) B_{m-k} X_{k+1}$. At equation (9) after substituting $\tilde{A}(t)X(t)$ and $B(t)X'^{(t)}$ with the respective expressions in (11) and (12) we get

$$\sum_{m \geq 0} C_m t^m + \sum_{m \geq 0} M_m t^m + \sum_{m \geq 0} Y_m t^m = 0 \qquad (13)$$

This density holds only if

$$C_m + M_m + Y_m = 0 \quad m \geq 0$$

or if

$$\sum_{k=0}^{m} (k+1) B_{m-k} X_{k+1} + \sum_{k=0}^{m} \breve{A}_{m-k} X_k + Y_m = 0 \quad m \geq 0.$$

Using the known initial condition $X_0 = X(0)$, the above equation for X_{m+1} is written

$$(m+1)B_0X_{m+1}+\sum_{k=0}^{m-1}(k+1)B_{m-k}X_{k+1}+\sum_{k=0}^{m}\bar{A}_{m-k}X_k+Y_m=0 \quad m\geq 0 (14)$$

This is a linear algebraic equation for X_{m+1} of the type $BX=b$ that can be solved using Moore -Penrose pseudoinverse. So, a solution of (14) has the form

$$X_{m+1}=\frac{-B_0^+}{m+1}\left[\sum_{k=0}^{m-1}(k+1)B_{m-k}X_{k+1}+\sum_{k=0}^{m-1}\bar{A}_{m-k}X_k+Y_m\right]m\geq 1_n \ (15)$$

To prove the compatibility of the system we must show the convergence of (10). From

$$\|\bar{A}_m\|\leq\frac{M}{r^m}, \|B_m\|\leq\frac{M}{r^m}, Y_m\leq\frac{M}{r^m}, 0<r<r_0, m\geq 0 \text{ for } M>1 (16)$$

From (15) taking the norms and using (16) it follows that

$$\|X_m\|\leq\frac{\|B_0^+\|M}{m}\left[\frac{\left(\sum_{k=0}^{m-2}(k+1)\|X_{k+1}\|r^k+\sum_{k=0}^{m-1}\|X_k\|r^k\right)+1}{r^{m-1}}\right]$$

Since we have the initial condition $X(0)=X_0$ we take $C_0=\|X_0\|$ and for $m=0$

$$B_0X_1=-Y_0+(I-A_0)X_0$$

and

$$X_1=B_0^+(-Y_0+(I-A_0)X_0)$$

so

$$\|X_1\|\leq\|B_0^+\|M(1+C_0)=C_1$$

From induction we get

$$C_m=\frac{\|B_0^+\|M}{m}\left[\frac{\left(\sum_{k=0}^{m-2}(k+1)C_{k+1}r^k\right)+\left(\sum_{k=0}^{m-1}C_kr^k\right)+1}{r^{m-1}}\right]$$

So $\|X_m\|\leq C_m$, $m\geq 0$.

We emphasize that B_0^+ is Penrose inverse matrix and for more you can see Israel and Greville, (2003) and Rakha, (2004).

3. Application

The data of the 35 sectors of Albanian economy are grouped in three big sectors which are agriculture, manufacturing and services. The INSTAT data are studied for a ten-year period (2009–2018). We use the input/output tables in millions of ALL.

As an initial time $(t=0)$ we use the data for year 2009. According to the information provided for 2009, the capital coefficient matrix and coefficient matrix for this model's coefficients for the Albanian economy are as follows:

Table 1 Coefficient matrix

	Coefficient matrix 2009			
	Agriculture	Manufacturing	Services	Demand vector
Agriculture	0.00	0.00	0.00	171940.628
Manufacturing	0.11	0.89	0.59	481683.8785
Services	0.08	0.36	0.87	690636.1794

Table 2 Capital coefficient matrix

	Capital Coefficient matrix 2009		
	Agriculture	Manufacturing	Services
Agriculture	0.00	0.00	0.00
Manufacturing	0.11	0.89	0.59
Services	0.08	0.36	0.87

The table 3 represents the solutions obtained for the consecutive years from 2009 to 2018 using MATLAB.

Table 3 Results with two methods

	Agriculture	Manufacturing	Services
X(0)	171940.628	481683.8785	690636.2
X(1)	125.124965	311.5283712	166.9375
X(2)	703.985918	2804.510511	264.6041
X(3)	3835.00084	24910.14373	-1815.18
X(4)	20224.898	202682.0285	-27370.9
X(5)	104267.685	1574254.106	-253526
X(6)	529659.891	11953659.03	-2063042
X(7)	2666596.14	89757965.49	-1.60E+07
X(8)	13366813.2	670184472.7	-1.20E+08
X(9)	66989447.2	4989436562	-9.00E+08

a) Algebraic method

	Agriculture	Manufacturing	Services
X(0)	171940.6	481683.9	690636.2
X(1)	342586	-451008	714993.3
X(2)	373264.9	-808607	299379.9
X(3)	-20853.7	34617.6	-128248
X(4)	-308942	453621.5	-183688
X(5)	-27320.7	-63480.9	-70759.1
X(6)	163353.2	-167748	57121.47
X(7)	-109902	159204.7	-6914.01
X(8)	-106543	33389.54	-66239.5
X(9)	119932	-113919	12919.36

b) Analytic method

The solution obtained from two methods, are interpreted in economy in the following tables

Table 4 The solution with Algebraic method

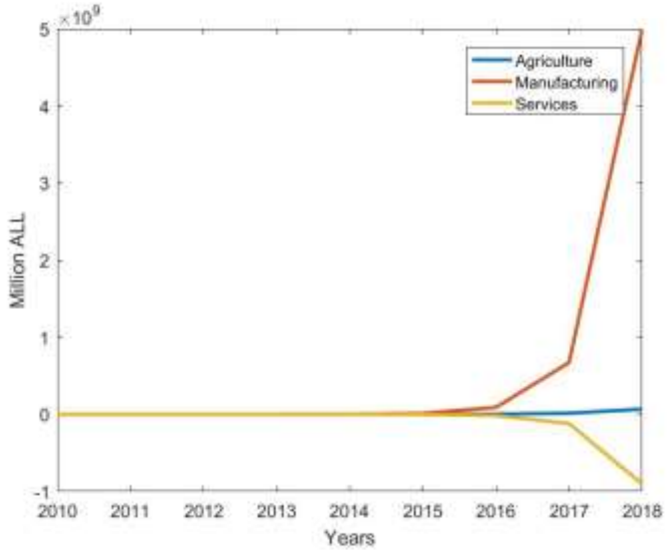
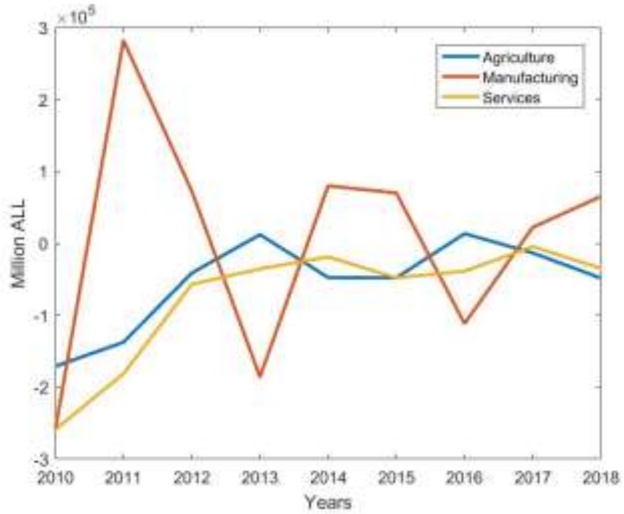
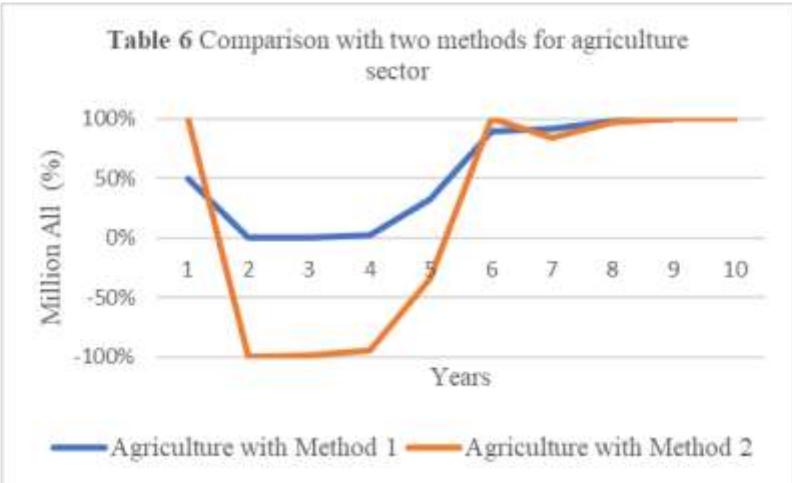


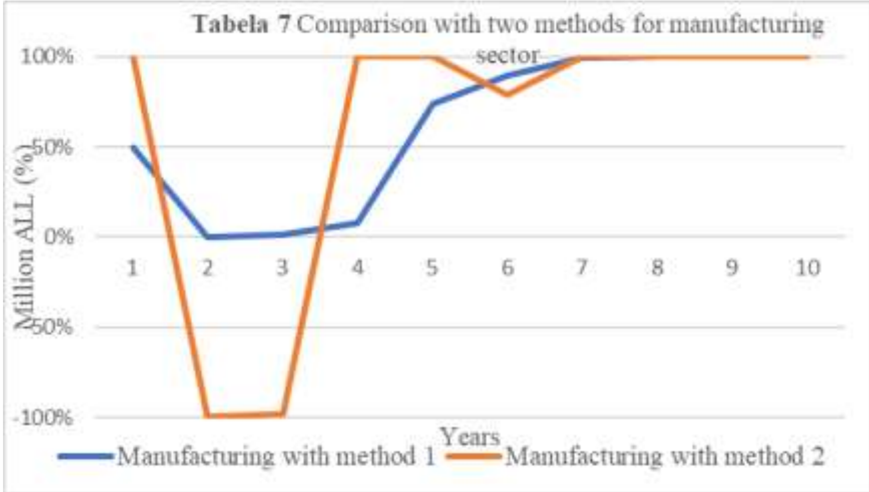
Table 5 The solution with Analytic method



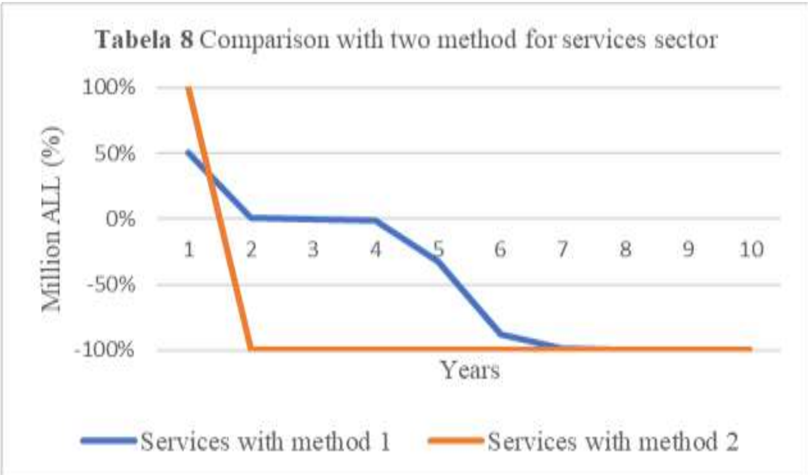
In the following tables we present the relation between the same sectors which we find with two methods**.** The values from table 3 are placed on the axis of ordinates, in percentages according to sectors, and the years where 1-10 means 2099-2018 are placed on the axis of ordinates.



The graph of table 6 shows that with method 1, the agricultural sector has a decline during the years 2009-2010, then for another 2 years it remains at the same level and in 2013 a significant increase begins. While with method 2 during the first 2 years the decrease is greater and continues constant for another two years, after which it begins to increase.



The graph of table 7 shows that with method 1, the manufacturing sector has a decline during the years 2009-2010, then for 1 year it remains at the same level and in 2011 a significant increase begins. While with method 2, during the first 2 years, the decrease is greater and faster and continues constant for another 1 year, then the increase begins, quickly decreases again in 2013 with the same rate and the increase begins.



The graph of table 8 shows that with method 1, the services sector has a decline during the years 2009-2010, then for 2 years it remains at the same level and in 2012 a significant decline begins.

Conclusion

The Leontief dynamical systems are used to study the growth of the gross national product and national income. These systems can be solved using the theory of matrix algebra and differential equations. In this work we studied the performance of Albanian economy for a ten-year period 2009-2018. We used the algebraic and analytic method and we evaluated which method gave the best solution for the Leontief dynamic model. For this, we regrouped the data of the three main sectors, then we obtained the solutions of the model with the two presented methods and we compared them graphically. Based on the analysis of the data and graphs solution given in section 3 of this paper, we suggest that the use of the algebraic method is more efficient than the analytic one. We consider it to be more useful for the analysis of different sectors because the economic ups and downs should not be rapidly precipitating because of the impact economic has in society and other sectors.

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