MODELLING TO COVID-19 DATA IN ALBANIA USING: AN EXTENDED BURR-TYPE X DISTRIBUTION

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Abstract

This article introduced a new distribution called the Marshall-Olkin Topp-Leone Half-Logistic-Burr type X, using the Marshall-Olkin Topp-Leone Half-Logistic-G family of distributions. We deduce some of its statistical properties. Choosing the most optimal estimators is one of the fundamental concerns in the theory of parameter estimation. We use several estimation approaches, including maximum likelihood estimation, least squares estimation, weighted least estimation, L-moment estimation, Maximum Product Spacing estimation, and minimal distance procedures, to estimate the parameters for the distribution. We will analyze simulation studies that evaluate the efficiency levels of different estimators using the Kolmogorov-Smirnov test. Finally, we conduct an analysis on a real COVID-19 data set in Albania to showcase the flexibility of our proposed model in contrast to the accuracy achieved by other alternative distributions.

Key words**:** *Burr type X distribution, Statistical Properties, Kolmogorov-Smirnov test, COVID-19*

Përmbledhje

Në këtë punim, do të tregojmë një shpërndarje të zgjeruar e emërtuar Marshall-Olkin Topp-Leone Half-Logistic-Burr të llojit X, e cila rrjedh nga një familje shpërndarjeje Marshall-Olkin Topp-Leone Half-Logistic-G. Për këtë shpërndarje do të tregohen disa veti statistikore. Në teorinë e vlerësuesve pikësorë zgjedhja e një vlerësuesi më të mirë ështe një problem kryesor. Për të vlerësuar parametrat e panjohur të shpërndarjes Marshall-Olkin Topp-Leone

Half-Logistic-Burr të llojit X do të trajtojmë disa metoda vlerësimi duke përfshirë, metodën e përgjagjësisë maksimale, metodën e katrorëve më të vegjël, metodën e katrorëve më të vegjël sipas peshave, vlerësuesit e L-momenteve, vlerësuesit e prodhimit të distancave maksimale, vlerësuesit e distancave minimale. Në do të analizojmë shkallën e efikasitet të secilave prej metoda të mësipërme duke u mbeshtetur në testin e Kolmogorov-Smirnov-it. Në fund, do të analizojmë disa të dhëna reale të COVID-19 në Shqipëri për të treguar fleksibilitetin e modelit tonë kundrejt disa shpërdarjeve të tjera alternative.

Fjalë kyçe: *Burr e llojit X, Veti Statistikore, Testi Kolmogorov-Smirnov, COVID-19*

1. Introduction

Burr (Irving, 1942) introduced twelve different forms of cumulative distribution functions for modelling data. Among those twelve distribution functions, Burr-Type X and Burr-Type XII received the maximum attention. There is a thorough analysis of Burr-Type XII distribution (Rodriguez, 1977), see also (Wingo, 1993) for a nice account of it.

In this paper, we consider the two-parameter Burr-Type X distribution. Surles and Padgett, (Surles and Padgett, 2001) proposed a new extension for Burr type X one parameter by adding scale parameter named Burr type X with two parameters or Burr type X distribution. Many authors have studied Burr type X distribution widely and applied them in different areas such as, (Merovci *et al*., 2016), (Raqab and Kundu, 2006), (Shayib and Haghighi, 2011), (Khaleel *et al*., 2016) and many others.

Definition 1.1 A random variable X has a two-parameter Burr-Type X distribution if cumulative distribution function (CDF) is given by:

$$F(x:\alpha,\theta) = \left(1 - e^{-(\theta x)^2}\right)^{\alpha};\ x > 0, \alpha > 0, \theta > 0 \quad (1)$$

where α and β are shape and scale parameters. Also, the probability density function (pdf) of the Burr type X distribution corresponding to the (1) is:

$$f(x:\alpha,\theta) = 2\alpha\theta^2 x e^{-(\theta x)^2}\left(1 - e^{-(\theta x)^2}\right)^{\alpha-1}; \qquad x > 0, \qquad \alpha > 0, \qquad \theta > 0 \quad (2)$$

The need for more flexible models that can explain and provide a better fit to real life data sets has motivated many researchers to develop new generalized distributions by extending the classical ones. Marshall and Olkin, (Marshall and Olkin, 1997) introduced a new method of adding a parameter into a family of distributions. Addition of parameters to an existing baseline distribution has proven to be an effective technique for improving the flexibility of new families of distributions (Barreto-Souza *et al.*, 2013). The Marshall-Olkin technique has been applied to build several well-known distributions which include papers by, (Cordeiro *et al.*, 2011), (Santos-Neo *et al.*, 2014), (Chakraborty *et al.*, 2017), and many others.

In 2023, Sengweni et al. introduced a generator of a continuous distribution called the Marshall-Olkin Topp-Leone Half-Logistic-G (MO-TLHL-G) family of distributions with pdf and cdf given by:

$$f(x:b,\delta,\varphi)=\frac{4b\delta g(x:\varphi)\bar{G}(x:\varphi)\left(1-\bar{G}^2(x:\varphi)\right)^{b-1}}{\left(1+\left(1-[1-\bar{G}^2(x:\varphi)]^b\right)\right)^2\left\{1-(1-\delta)\left(1-\left[\frac{[1-\bar{G}^2(x:\varphi)]^b}{1+\left(1-[1-\bar{G}^2(x:\varphi)]^b\right)}\right]\right)\right\}^2} \quad (3)$$

and

$$F(x:b,\delta,\varphi)=\frac{\frac{[1-\bar{G}^2(x:\varphi)]^b}{\left(1+\left(1-[1-\bar{G}^2(x:\varphi)]^b\right)\right)}}{1-(1-\delta)\left(1-\left[\frac{[1-\bar{G}^2(x:\varphi)]^b}{1+\left(1-[1-\bar{G}^2(x:\varphi)]^b\right)}\right]\right)} \quad (4)$$

for $\delta, b>0$ and φ is a parameter vector for the baseline distribution $G(\cdot)$.

In this paper, we propose a new extension of Burr type X distribution with more flexibility than the baseline (1) and (2). The new distribution can be applied to different kinds of data because the four shape parameters can control the tail of data. The new distribution has more sub-models when compared with baseline distribution and hence it allows us to study more comprehensive structural properties. Thus, the aims of this work are to explore and study the mathematical properties of the new distribution, which is an extended Burr type X distribution and to prove the new model is more flexible than other models by applying it to real data using a goodness of fit test for real data.

Definition 1.2 A random variable X is said to have a Marshall-Olkin Topp-Leone Half-Logistic- Burr type X distribution with a vector parameter (b,δ,α,θ) if its probability density function is defined as:

$$f(x:b,\delta,\alpha,\theta)=\frac{8b\delta\alpha\theta^{2}xe^{-(\theta x)^{2}}\left(1-e^{-(\theta x)^{2}}\right)^{\alpha-1}\left(1-\left(1-e^{-(\theta x)^{2}}\right)^{\alpha}\right)\left(\left(1-\left(1-e^{-(\theta x)^{2}}\right)^{\alpha}\right)^{2}\right)^{b-1}}{\left(1+\left(1-\left[1-\left(1-\left(1-e^{-(\theta x)^{2}}\right)^{\alpha}\right)^{2}\right]^{b}\right)\right)^{2}\left\{1-(1-\delta)\left(1-\left[\frac{\left[1-\left(1-\left(1-e^{-(\theta x)^{2}}\right)^{\alpha}\right)^{2}\right]^{b}}{1+\left(1-\left[\left(1-\left(1-e^{-(\theta x)^{2}}\right)^{\alpha}\right)^{2}\right]^{b}\right)}\right]\right)\right\}^{2}} \tag{5}$$

and cumulative distribution function

$$F(x:b,\delta,\alpha,\theta)=\frac{\dfrac{\left[1-\left(1-\left(1-e^{-(\theta x)^{2}}\right)^{\alpha}\right)^{2}\right]^{b}}{\left(1+\left(1-\left[1-\left(1-\left(1-e^{-(\theta x)^{2}}\right)^{\alpha}\right)^{2}\right]^{b}\right)\right)}}{1-(1-\delta)\left(1-\left[\dfrac{\left[1-\left(1-\left(1-e^{-(\theta x)^{2}}\right)^{\alpha}\right)^{2}\right]^{b}}{1+\left(1-\left[1-\left(1-\left(1-e^{-(\theta x)^{2}}\right)^{\alpha}\right)^{2}\right]^{b}\right)}\right]\right)} \tag{6}$$

respectively, for $b>0,\delta>0,\alpha>0,\theta>0$.

Figures 1 and Figures 2 illustrate some of the possible shapes of the(pdf) and (cdf) of the Marshall-Olkin Topp-Leone Half-Logistic- Burr type X (MO-TLHL-BX) distribution for selected values of the parameters b,δ,α and θ.

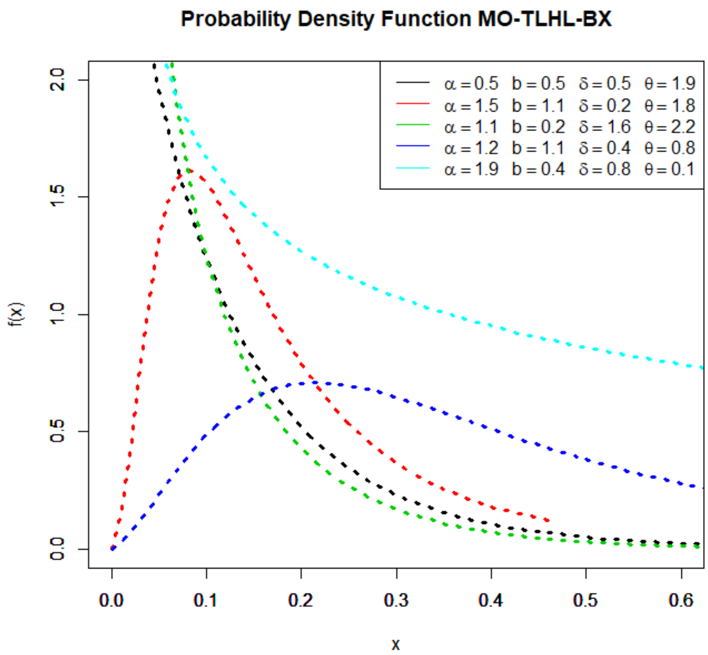

Figure 1. Probability Density Function (pdf) of the MO-TLHL-BX.

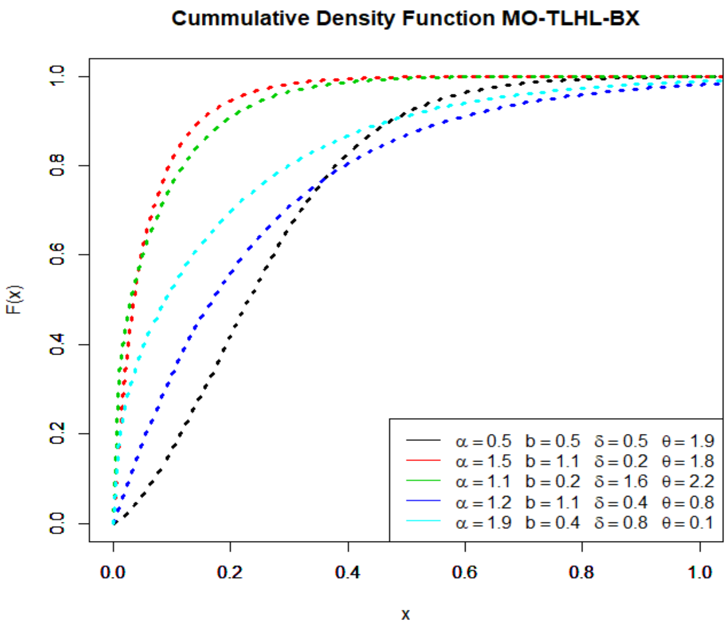

Figure 2. Cumulative Density Function(cdf) of the MO-TLHL-BX.

2 Mathematical Properties

2.1 Sub-models

It is worth noting that many sub-models can be obtained as special cases of the MO-TLHL-BX distribution by selecting specific values of parameters in equation (5).

If $\delta = 1$, we obtain a new family of distributions called the Topp-Leone Half-Logistic- BX (TLHL-G) distribution with pdf given by:

$$f(x: b, \delta = 1, \alpha, \theta) = 2b2\alpha\theta^2 x e^{-(\theta x)^2}\left(1 - e^{-(\theta x)^2}\right)^{\alpha-1} \times \left(1 - \left(1 - \left(1 - e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right)^{b-1} \times exp\left[-\left(\frac{\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]^b}{1-\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]^b}\right)\right] \quad (7)$$

If $\delta = b = 1$, we obtain a new family of distributions with pdf given by:

$$f(x: b = 1, \delta, \alpha, \theta) = 4\alpha\theta^2 x e^{-(\theta x)^2} exp\left[-\left(\frac{1-\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]}{\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]}\right)\right] \quad (8)$$

If $b = 1$, we obtain a new family of distributions with pdf given by:

$$f(x: b = 1, \delta, \alpha, \theta) = \frac{2\delta 2\alpha\theta^2 x e^{-(\theta x)^2} exp\left[-\left(\frac{1-\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]}{\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]}\right)\right]}{\left[1-(1-\delta) exp\left[-\left(\frac{1-\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]}{\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]}\right)\right]\right]^2} \quad (9)$$

2.2 Survival Function

The reliability function (survival function) of the Marshall-Olkin Topp-Leone Half-Logistic- Burr type X (MO-TLHL-BX) distribution is given by:

$$R(x;b,\delta,\alpha,\theta)=1-\frac{\dfrac{\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]^b}{\left(1+\left(1-\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]^b\right)\right)}}{1-(1-\delta)\left(1-\left[\dfrac{\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]^b}{1+\left(1-\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]^b\right)}\right]\right)} \tag{10}$$

2.3 Hazard Function

The hazard rate function (failure rate) of the Marshall-Olkin Topp-Leone Half-Logistic- Burr type X (MO-TLHL-BX) distribution is given by:

$h(x;b,\delta,\alpha,\theta)=$

$$\frac{\dfrac{8b\delta\alpha\theta^2 x e^{-(\theta x)^2}\left(1-e^{-(\theta x)^2}\right)^{\alpha-1}\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)\left(\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right)^{b-1}}{\left(1+\left(1-\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]^b\right)\right)^2\left\{1-(1-\delta)\left(1-\left[\dfrac{\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]^b}{1+\left(1-\left[\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]^b\right)}\right]\right)\right\}^2}}{1-\dfrac{\dfrac{\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]^b}{\left(1+\left(1-\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]^b\right)\right)}}{1-(1-\delta)\left(1-\left[\dfrac{\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]^b}{1+\left(1-\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]^b\right)}\right]\right)}} \tag{11}$$

2.4 Some Useful Expression

In this section, expansion for the pdf of the MO-TLHL-BX distribution is derived, which are useful to study several statistical properties of MO-TLHL-BX distribution. The generalized binomial is used to find the expression of the pdf.

$$f(x\colon b,\delta,\alpha,\theta)=8b\delta\alpha\theta^2 x e^{-(\theta x)^2}\left(1-e^{-(\theta x)^2}\right)^{\alpha-1}\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)\times$$

$$\left(1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^{b-1}\right)\left(1+\left(1-\left(1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^{2}\right)^{b}\right)\right)^{-2}\times$$

$$\underbrace{\left\{1-(1-\delta)\left(1-\left[\frac{\left(1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^{2}\right)^{b}}{1+\left(1-\left(1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^{2}\right)^{b}\right)}\right]\right)\right\}^{-2}} \to a \qquad (12)$$

Using the generalized binomial series expansion for a, we have:

$$a=\sum_{l,k=0}^{\infty}(-1)^{l+k}\binom{-2}{l}\binom{l}{k}(1-\delta)^{l}\frac{\left(1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^{2}\right)^{bk}}{\left[1+\left(1-\left(1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^{2}\right)^{b}\right)\right]^{k}}$$

Substituting a in equation (12) we have:

$$f(x:b,\delta,\alpha,\theta)=8b\delta\alpha\theta^{2}xe^{-(\theta x)^2}\left(1-e^{-(\theta x)^2}\right)^{\alpha-1}\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)\sum_{l,k=0}^{\infty}(-1)^{l+k}\binom{-2}{l}\binom{l}{k}(1-\delta)^{l}\left(1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^{2}\right)^{b(k+1)-1}\times\underbrace{\left[1+\left(1-\left(1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^{2}\right)^{b}\right)\right]^{-(k+2)}}\to b \qquad (13)$$

Applying the generalized binomial series expansion on b, we have:

$$b=\sum_{i,j=0}^{\infty}(-1)^{i}\binom{-(k+2)}{j}\binom{j}{i}\left(1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^{2}\right)^{bi}$$

Substituting b in equation (13), we have:

$$f(x:b,\delta,\alpha,\theta)=8b\delta\alpha\theta^{2}xe^{-(\theta x)^2}\left(1-e^{-(\theta x)^2}\right)^{\alpha-1}\times$$

$$\times\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)\sum_{l,k=0}^{\infty}(-1)^{l+k}\binom{-2}{l}\binom{-(k+2)}{j}\binom{j}{i}\binom{l}{k}(1-\delta)^{l}\times$$

$$\underbrace{\left(1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^{2}\right)^{b(k+i+1)-1}}\to c \qquad (14)$$

Applying the generalized binomial series expansion on c, we have:

$$c=\sum_{q=0}^{\infty}(-1)^{q}\binom{b(k+i+1)-1}{q}\left(\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)\right)^{2q}$$

Substituting c in equation (14) we obtain:

$$f(x\colon b,\delta,\alpha,\theta)=8b\delta\alpha\theta^{2}xe^{-(\theta x)^{2}}$$

$$\times\left(1-e^{-(\theta x)^{2}}\right)^{\alpha-1}\sum_{l,k,i,j,q=0}^{\infty}(-1)^{l+k+i+q}\binom{-2}{l}\binom{-(k+2)}{j}\binom{j}{i}\binom{l}{k}\binom{b(k+i+1)-1}{q}$$

$$\times$$

$$\times(1-\delta)^{l}\left(\left(1-\left(1-e^{-(\theta x)^{2}}\right)^{\alpha}\right)\right)^{2q+1} \tag{15}$$

Using the binomial series expansion, we have

$$\left(\left(1-\left(1-e^{-(\theta x)^{2}}\right)^{\alpha}\right)\right)^{2q+1}=\sum_{p=0}^{\infty}(-1)^{p}\binom{2q+1}{p}\left(\left(1-e^{-(\theta x)^{2}}\right)^{\alpha}\right)^{p}$$

So, the probability density function of the MO-TLHL-BX distribution can be expressed as:

$$f(x\colon b,\delta,\alpha,\theta)=8b\delta\alpha\theta^{2}xe^{-(\theta x)^{2}}\times\left(1-e^{-(\theta x)^{2}}\right)^{\alpha-1}\times$$

$$\sum_{l,k,i,j,q=0}^{\infty}(-1)^{l+k+i+q+p}\binom{-2}{l}\binom{-(k+2)}{j}\binom{j}{i}\binom{l}{k}\binom{b(k+i+1)-1}{q}\times$$

$$\binom{2q+1}{p}(1-\delta)^{l}\left(\left(1-e^{-(\theta x)^{2}}\right)^{\alpha}\right)^{p} \tag{16}$$

2.5 Order Statistics

For $X_1, X_2, \ldots, X_n$ i.i.d. continous random variables with pdf (5) and cdf (6) the density of the maximum order is

$$p_{(n)}(x)=np(x)F(x)^{n-1}=$$

$$n\left(\frac{8b\delta\alpha\theta^{2}xe^{-(\theta x)^{2}}\left(1-e^{-(\theta x)^{2}}\right)^{\alpha-1}\left(1-\left(1-e^{-(\theta x)^{2}}\right)^{\alpha}\right)\left(\left(1-\left(1-e^{-(\theta x)^{2}}\right)^{\alpha}\right)^{2}\right)^{b-1}}{\left(1+\left(1-\left[1-\left(1-\left(1-e^{-(\theta x)^{2}}\right)^{\alpha}\right)^{2}\right]^{b}\right)\right)^{2}\left\{1-(1-\delta)\left(1-\left[\frac{\left[1-\left(1-\left(1-e^{-(\theta x)^{2}}\right)^{\alpha}\right)^{2}\right]^{b}}{1+\left(1-\left[\left(1-\left(1-e^{-(\theta x)^{2}}\right)^{\alpha}\right)^{2}\right]^{b}\right)}\right]\right)\right\}^{2}}\right)\times$$

$$\left(\frac{\dfrac{\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]^b}{\left(1+\left(1-\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]^b\right)\right)}}{1-(1-\delta)\left(1-\left[\dfrac{\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]^b}{1+\left(1-\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]^b\right)}\right]\right)}\right)^{n-1} \tag{17}$$

For $X_1, X_2, \ldots, X_n$ iid continous random variables with pdf (5) and cdf (6) the density of the minimum order is

$$p_{(1)}(x) = np(x)\left(1-F(x)\right)^{n-1} =$$

$$n\left(\frac{8b\delta\alpha\theta^2 x e^{-(\theta x)^2}\left(1-e^{-(\theta x)^2}\right)^{\alpha-1}\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)\left(\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right)^{b-1}}{\left(1+\left(1-\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]^b\right)\right)^2\left\{1-(1-\delta)\left(1-\left[\dfrac{\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]^b}{1+\left(1-\left[\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]^b\right)}\right]\right)\right\}^2}\right)\times$$

$$\left(1-\frac{\dfrac{\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]^b}{\left(1+\left(1-\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]^b\right)\right)}}{1-(1-\delta)\left(1-\left[\dfrac{\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]^b}{1+\left(1-\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]^b\right)}\right]\right)}\right)^{n-1} \tag{18}$$

For $X_1, X_2, \ldots, X_n$ iid continous random variables with pdf (5) and cdf (6) the density of the *k*th order is:

$$p_{(k)}(x) = np(x)\binom{n-1}{k-1}F(x)^{k-1}\big(1-F(x)\big)^{n-k} =$$

$$\binom{n-1}{k-1}\left(\frac{8b\delta\alpha\theta^2 x e^{-(\theta x)^2}\left(1-e^{-(\theta x)^2}\right)^{\alpha-1}\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)\left(\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right)^{b-1}}{\left(1+\left(1-\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]^b\right)\right)^2\left\{1-(1-\delta)\left(1-\left[\frac{\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]^b}{1+\left(1-\left[\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]^b\right)}\right]\right)\right\}^2}\right)\times$$

$$\times\left(1-\frac{\dfrac{\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]^b}{\left(1+\left(1-\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]^b\right)\right)}}{1-(1-\delta)\left(1-\left[\dfrac{\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]^b}{1+\left(1-\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]^b\right)}\right]\right)}\right)^{n-k}\times$$

$$\left(\frac{\dfrac{\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]^b}{\left(1+\left(1-\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]^b\right)\right)}}{1-(1-\delta)\left(1-\left[\dfrac{\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]^b}{1+\left(1-\left[1-\left(1-\left(1-e^{-(\theta x)^2}\right)^{\alpha}\right)^2\right]^b\right)}\right]\right)}\right)^{k-1} \quad (19)$$

2.6 Rényi Entropy

Rényi entropy is a quantity utilized in information theory that serves to generalize several entropy concepts, such as Hartley entropy, Shannon entropy, collision entropy, and mine-entropy. Rényi entropy, as proposed by Alfréd Rényi in 1960, derives its name from the scientist who sought to identify the most general method of quantifying information while maintaining additivity for independent events. The notion of generalized dimensions is founded upon the Rényi entropy within the domain of fractal dimension estimation. Entropy as a Rényi value is significant in statistics and ecology as a measure of diversity. Additionally significant in quantum information, the Rényi entropy can be utilized as a quantification of entanglement. The explicit calculation of the Rényi entropy as a function of α is feasible in the Heisenberg XY spin chain model due to the automorphic nature of the function with respect to a specific subgroup of the modular group. As it relates to random extractors, min-entropy is a term used in theoretical computer science.

Rényi entropy $I_R(\nu)$ for the Marshall-Olkin Topp-Leone Half-Logistic- Burr type X (MO-TLHL-BX) distribution is given by:

$$I_R(\nu) = (1-\nu)^{-1} log\left[\int_0^\infty f^\nu(x)dx\right] = (1-\nu)^{-1} \times$$

$$\times \log\left[\sum_{p=0}^{\infty}\sum_{l,k,i,j,q=0}^{\infty}(-1)^{l+k+i+q+p}(4b\delta)^\nu\binom{-2\nu}{l}\binom{l}{k}\binom{-(2\nu+k)}{j}\binom{j}{i}\binom{2q+\nu}{p}\right.$$

$$\left.\binom{b(k+i+v)-\nu}{q}(1-\delta)^l\left(\frac{p}{\nu}+1\right)\int_0^\infty\left(2\alpha\theta^2 x e^{-(\theta x)^2}\left(1-e^{-(\theta x)^2}\right)^{\alpha-1}\left(\left(1-e^{-(\theta x)^2}\right)^\alpha\right)^{\frac{p}{\nu}}\right)^\nu dx\right] \quad (20)$$

3. Methods for Estimating Parameters

In this section, the unknown parameters of the MO-TLHL-BX distribution are estimated using different methods of estimation.

3.1 Maximum Likelihood Estimation

The ML method, also called full information maximum likelihood, is most widely used because it generates estimates with highly desirable large sample properties. These properties also approximately hold in finite samples. For linear models with normally distributed errors, the ML estimator (MLE) is unbiased, normally distributed, and most efficient.

Clearly, given a Φ, $L(\Phi)$ represents the probability for the sample to be observed. Because the sample is already observed, the idea of ML is to find a value of Φ that maximizes this probability. Formally, the MLE is defined by the value $\widetilde{\Phi}$ that maximizes $L(\Phi)$.

In our case, $x_1, x_2, \dots, x_n$ be i.i.d. random variables with a a probability density function (5). The likelihood function of parameters $b, \delta, \alpha, \theta$ is:

$$\ell = nln(4b) + nln(\delta) + \sum_{i=1}^{n} ln\left(2\alpha\theta^2 x e^{-(\theta x)^2}\left(1 - e^{-(\theta x)^2}\right)^{\alpha-1}\right) + \sum_{i=1}^{n} ln\left(1 - \left(1 - e^{-(\theta x)^2}\right)^{\alpha}\right) + (b-1) \times \sum_{i=1}^{n} ln\left(1 - 1 - \left(1 - e^{-(\theta x)^2}\right)^{\alpha}\right) - 2\sum_{i=1}^{n} ln\left(1 + \left(1 - \left(1 - \left(1 - \left(1 - e^{-(\theta x)^2}\right)^{\alpha}\right)\right)\right)^{b}\right) - -2\sum_{i=1}^{n} ln\left(1 - (1 - \delta)\left(\frac{\left(1-\left(1-\left(1-e^{-(\theta x_i)^2}\right)^{\alpha}\right)^2\right)^{b}}{1+\left(1-\left(1-\left(1-\left(1-e^{-(\theta x_i)^2}\right)^{\alpha}\right)\right)\right)^{b}}\right)\right) \quad (21)$$

The exact solution for unknown parameters is not possible analytically, so estimates are obtained by solving nonlinear equations simultaneously. The solution of nonlinear systems is easier with iterative techniques such as the Newton-Raphson approach. By providing an initial guess of the parameters, Newton Raphson used these initial values to calculate parameter estimates. Asymptotically, these estimates of parameters approach normality, and the z-score is approximately standard normal, which can be used to find the $100(1-\alpha)$ two-sided confidence interval for the parameters.

3.2 Least Square Estimation

The least square estimators and weighted least square estimators (LSEs) were proposed by Swain, Venkatraman and Wilson (Swain (1988)) to estimate the parameters of a Beta distribution.

The LSEs of the unknown parameters of MO-TLHL-BX distribution can be obtained by minimizing:

$$\sum_{j=1}^{n}\left(F\left(x_{(j)}; b, \delta, \alpha, \theta\right) - \frac{j}{n+1}\right)^2 \quad (22)$$

with respect to the unknown parameters $b, \delta, \alpha, \theta$. Where $F(\cdot)$ denotes the distribution function of the MO-TLHL-BX distribution and $E\left(F\left(X_{(j)}\right)\right) =$

$\frac{j}{n+1}$ is the expectation of the empirical cumulative distribution function. The least squares estimate (LSEs) of b,δ,α,θ, say, $\hat{b}_{LSE},\hat{\delta}_{LSE},\hat{\alpha}_{LSE},\hat{\theta}_{LSE}$ respectively, can be obtained by minimizing:

$$LS(x_j;b,\delta,\alpha,\theta)=\sum_{j=1}^{n}\left(\frac{\dfrac{\left[1-\left(1-\left(1-e^{-(\theta x_j)^2}\right)^{\alpha}\right)^{2}\right]^{b}}{\left(1+\left(1-\left[1-\left(1-\left(1-e^{-(\theta x_j)^2}\right)^{\alpha}\right)^{2}\right]^{b}\right)\right)}}{1-(1-\delta)\left(1-\left[\dfrac{\left[1-\left(1-\left(1-e^{-(\theta x_j)^2}\right)^{\alpha}\right)^{2}\right]^{b}}{1+\left(1-\left[1-\left(1-\left(1-e^{-(\theta x_j)^2}\right)^{\alpha}\right)^{2}\right]^{b}\right)}\right]\right)}-\frac{j}{n+1}\right)^{2}$$

Therefore, $\hat{b}_{LSE},\hat{\delta}_{LSE},\hat{\alpha}_{LSE},\hat{\theta}_{LSE}$ of b,δ,α,θ can be obtained as the solution of the following system of equations:

$$\frac{\partial LS(x_j;b,\delta,\alpha,\theta)}{\partial b}=0,\ \frac{\partial LS(x_j;b,\delta,\alpha,\theta)}{\partial \delta}=0,\ \frac{\partial LS(x_j;b,\delta,\alpha,\theta)}{\partial \alpha}=0,\ \frac{\partial LS(x_j;b,\delta,\alpha,\theta)}{\partial \theta}=0$$

We can solve these equations numerically to obtain the estimates $\hat{b}_{LSE},\hat{\delta}_{LSE},\hat{\alpha}_{LSE},\hat{\theta}_{LSE}$.

3.3 The Weighted Least Square Estimation

The weighted least squares estimators (WLSEs) of the unknown parameters can be obtained by minimizing:

$$\sum_{j=1}^{n}\omega_j\left(F(x_{(j)})-\frac{j}{n+1}\right)^2 \tag{23}$$

with respect to α,β,θ , where ω_j denotes the weight function at the jth point, which is equal to

$$\omega_j=\frac{1}{V(F(X_{(j)})}\frac{(n+1)^2(n+2)}{j(n-j+1)}$$

The weighted least square estimates (WLSEs) say $\hat{b}_{WLSE},\hat{\delta}_{WLSE},\hat{\alpha}_{WLSE}$ and $\hat{\theta}_{WLSE}$ can be obtained by minimizing:

$$WLS(x_j;b,\delta,\alpha,\theta)=$$

$$= \sum_{j=1}^{n} \frac{(n+1)^2(n+2)}{j(n-j+1)} \left(\frac{ \frac{\left[1-\left(1-\left(1-e^{-(\theta x_j)^2}\right)^{\alpha}\right)^2\right]^b}{\left(1+\left(1-\left[1-\left(1-\left(1-e^{-(\theta x_j)^2}\right)^{\alpha}\right)^2\right]^b\right)\right)} }{ 1-(1-\delta)\left(1-\left[\frac{\left[1-\left(1-\left(1-e^{-(\theta x_j)^2}\right)^{\alpha}\right)^2\right]^b}{\left(1+\left(1-\left[1-\left(1-\left(1-e^{-(\theta x_j)^2}\right)^{\alpha}\right)^2\right]^b\right)\right)}\right]\right) } - \frac{j}{n+1} \right)^2$$

Therefore, the estimators $\hat{b}_{WLSE}, \hat{\delta}_{WLSE}, \hat{\alpha}_{WLSE}, \hat{\theta}_{WLSE}$ can be obtained from the first partial derivative with respects to α, β, θ and set the result equal to zero:

$$\frac{\partial WLS(x_j;b,\delta,\alpha,\theta)}{\partial \alpha} = 0, \frac{\partial WLS(x_j;b,\delta,\alpha,\theta)}{\partial \beta} = 0, \frac{\partial WLS(x_j;b,\delta,\alpha,\theta)}{\partial \alpha} = 0, \frac{\partial WLS(x_j;b,\delta,\alpha,\theta)}{\partial \theta} = 0$$

By solving these equations numerically, we can obtain the estimates $\hat{b}_{WLSE}, \hat{\delta}_{WLSE}, \hat{\alpha}_{WLSE}$ and $\hat{\theta}_{WLSE}$.

3.4 L-Moments Estimators

The L-moments estimators were originally proposed by Hosking (1990). These estimators are obtained by equating the sample L-moments with the population L-moments. Hosking (1990) states that the L-moment estimators are more robust than the moment estimators and they are also relatively robust to the effects of outliers and reasonably efficient when compared to the maximum likelihood estimators for some distributions.

For the MO-TLHL-BX distribution, the L-moments estimators can be obtained by equating the first three sample L-moments with the corresponding population L-moments. The first three sample L-moments are:

$$l_1 = \frac{1}{n}\sum_{j=1}^{n} x_{(j)},$$

$$l_2 = \frac{2}{n(n-1)}\sum_{j=2}^{n} (j-1)x_{(j)} - l_1$$

$$l_3 = \frac{6}{n(n-1)(n-2)} \sum_{j=3}^{n} (j-1)(j-2)x_{(j)} - 6l_2 + l_1$$

and the first three population L-moments are:

$$\lambda_1 = E(X_{1:1}) = \int_{-\infty}^{+\infty} x\, f(x)dx = E(X),$$

$$\lambda_2 = \frac{1}{2}[E(X_{2:2}) - E(X_{2:1})] = \int_{-\infty}^{+\infty} x[2F(x) - 1]f(x)dx,$$

$$\lambda_2 = \frac{1}{3}[E(X_{3:3}) - 2E(X_{2:3}) + E(X_{1:3})] = \int_{-\infty}^{+\infty} x[6\big(F(x)\big)^2 - 6F(x) + 1]f(x)dx$$

Here, $X_{j:n}$ denotes the jth order statistic of a sample of size n. Therefore, the L-moments estimators $\hat{b}_{LME}, \hat{\delta}_{LME}, \hat{\alpha}_{LME}, \hat{\theta}_{LME}$ of the parameters $b, \delta, \alpha, \theta$ can be obtained by solving numerically the following equations:

$\lambda_1(\hat{\alpha}_{LME}, \hat{\beta}_{LME}, \hat{\theta}_{LME}) = l_1$, $\lambda_2(\hat{\alpha}_{LME}, \hat{\beta}_{LME}, \hat{\theta}_{LME}) = l_2$, $\lambda_3(\hat{\alpha}_{LME}, \hat{\beta}_{LME}, \hat{\theta}_{LME}) = l_3$

3.5 Maximum Product Spacing Estimators

The Maximum Product of Spacings (MPS) approach, which is used to estimate parameters in continuous univariate distributions, was first suggested by Cheng and Amin in 1983. Independently, Ranneby also developed this method in 1984 as an approximation of the Kullback-Leibler measure of information. This technique relies on the concept that the variations between successive sites should have the same distribution.

Let $X_1, X_2, \ldots, X_n$ be a random sample from the MO-TLHL-BX distribution and $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be an ordered random sample. For convenience, we also denote $X_0 = -\infty$ and $X_n = +\infty$. In the method of maximum product of spacings, we seek to estimate the parameters $b, \delta, \alpha, \theta$ of the distribution by maximizing the geometric mean of distances D_i, where every distance D_i is defined as

$$D_i = \int_{x_{(i-1)}}^{x_{(i)}} f(x;\theta)\, dx = F\big(x_{(i)}; b, \delta, \alpha, \theta\big) - F\big(x_{(i-1)}; b, \delta, \alpha, \theta\big) \quad \text{for } i = 1, 2, \ldots, n+1 \qquad (20)$$

where $F\big(x_{(0)}; b, \delta, \alpha, \theta\big) = 0$, $F\big(x_{(n+1)}; b, \delta, \alpha, \theta\big) = 1$ and $\sum_{i=1}^{n+1} D_i = 1$.

The geometric mean of distances is given by:

$$GM = \sqrt[n+1]{\prod_{i=1}^{n+1} D_i} \tag{24}$$

The MPS estimators $\hat{b}_{MPS}, \hat{\delta}_{MPS}, \hat{\alpha}_{MPS}, \hat{\theta}_{MPS}$ are obtained by maximizing the geometric mean (GM) of the spacings with respect to $b, \delta, \alpha, \theta$ or equivalently by maximizing the logarithm of the geometric mean of the sample spacings:

$$\log(GM) = \log\left(\sqrt[n+1]{\prod_{i=1}^{n+1} D_i}\right) = \frac{1}{n+1}\sum_{i=1}^{n+1} logD_i$$

$$= \frac{1}{n+1}\sum_{i=1}^{n+1} \log\left[F\left(x_{(i)}; b, \delta, \alpha, \theta\right) - F\left(x_{(i-1)}; b, \delta, \alpha, \theta\right)\right]$$

$$= \frac{1}{n+1}\sum_{i=1}^{n+1} \log\left[\frac{\dfrac{\left[1-\left(1-\left(1-e^{-(\theta x_i)^2}\right)^{\alpha}\right)^2\right]^b}{\left(1+\left(1-\left[1-\left(1-\left(1-e^{-(\theta x_i)^2}\right)^{\alpha}\right)^2\right]^b\right)\right)}}{1-(1-\delta)\left(1-\left[\dfrac{\left[1-\left(1-\left(1-e^{-(\theta x_i)^2}\right)^{\alpha}\right)^2\right]^b}{1+\left(1-\left[1-\left(1-\left(1-e^{-(\theta x_i)^2}\right)^{\alpha}\right)^2\right]^b\right)}\right]\right)} - \frac{\dfrac{\left[1-\left(1-\left(1-e^{-\left(\theta x_{(i-1)}\right)^2}\right)^{\alpha}\right)^2\right]^b}{\left(1+\left(1-\left[1-\left(1-\left(1-e^{-\left(\theta x_{(i-1)}\right)^2}\right)^{\alpha}\right)^2\right]^b\right)\right)}}{1-(1-\delta)\left(1-\left[\dfrac{\left[1-\left(1-\left(1-e^{-\left(\theta x_{(i-1)}\right)^2}\right)^{\alpha}\right)^2\right]^b}{1+\left(1-\left[1-\left(1-\left(1-e^{-\left(\theta x_{(i-1)}\right)^2}\right)^{\alpha}\right)^2\right]^b\right)}\right]\right)}\right]$$

3.6 Methods of Minimum Distances

The minimum distance method is a versatile approach that formalizes the inference issue by seeking a distribution function that closely approximates the empirical distribution derived from the observed data. The technique of minimal distance estimate was first introduced by Wolfmitz in 1950. The approach involves estimating the parameters of a distribution by minimizing the statistics of the empirical distribution function, commonly known as goodness-of-fit statistics. The minimal distance approach offers many estimators based on the selected empirical distribution function statistic. Within this part, we provide three estimate techniques for the MO-TLHL-BX distribution. These approaches include minimizing the goodness-of-fit statistics in relation to the parameters, b, δ, α and θ. This statistical approach relies on comparing the estimate of the cumulative distribution function with the empirical distribution function (see to D'Agostino, R. (1986) and Luceno, Alberto (2006)).

3.6.1 Method of Cramér-von-Mises

The Cramér-von-Mises estimator (CME) is a type of minimum distance estimator, which is based on the Cramér-von-Mises statistic (Cramér, H. (1928), Cramér von Mises, R. E. (1928)). MacDonald (1971) motivates the choice of Cramér-von-Mises type minimum distance estimators providing empirical evidence that the bias of the estimator is smaller than the other minimum distance estimators. The Cramér-von-Mises estimates $\hat{b}_{CME}, \hat{\delta}_{CME}, \hat{\alpha}_{CME}, \hat{\theta}_{CME}$ of parameters $b, \delta, \alpha, \theta$ of MO-TLHL-BX distribution are obtained by minimizing, with respect to b, δ, α and θ the function:

$$C(\alpha, \beta, \theta) = \frac{1}{12n} + \sum_{i=1}^{n} \left(F\left(x_{(i)} | b, \delta, \alpha, \theta\right) - \frac{2i-1}{n} \right)^2$$

$$C(b, \delta, \alpha, \theta) = \frac{1}{12n} + \sum_{i=1}^{n} \left(\frac{1 - exp\left(1 - \left(\left(\left(\frac{(\theta+1)x_i e^{\frac{\theta}{x_i}}}{\theta + \theta x_i + x_i} - 1\right)^{-\alpha} + 1\right)\right)^{\beta}\right)}{1 + exp\left(1 - \left(\left(\left(\frac{(\theta+1)x_i e^{\frac{\theta}{x_i}}}{\theta + \theta x_i + x_i} - 1\right)^{-\alpha} + 1\right)\right)^{\beta}\right)} - \frac{2i-1}{n} \right)^2 \tag{25}$$

These estimates can be obtained by solving the nonlinear equations:

$$\sum_{i=1}^{n} \left(F\left(x_{(i)} | b, \delta, \alpha, \theta\right) - \frac{2i-1}{n} \right)^2 \frac{\partial F\left(x_{(i)} | b, \delta, \alpha, \theta\right)}{\partial b} = 0$$

$$\sum_{i=1}^{n}\left(F\left(x_{(i)}|b,\delta,\alpha,\theta\right)-\frac{2i-1}{n}\right)^{2}\frac{\partial F\left(x_{(i)}|b,\delta,\alpha,\theta\right)}{\partial\delta}=0$$

$$\sum_{i=1}^{n}\left(F\left(x_{(i)}|b,\delta,\alpha,\theta\right)-\frac{2i-1}{n}\right)^{2}\frac{\partial F\left(x_{(i)}|b,\delta,\alpha,\theta\right)}{\partial\alpha}=0$$

$$\sum_{i=1}^{n}\left(F\left(x_{(i)}|b,\delta,\alpha,\theta\right)-\frac{2i-1}{n}\right)^{2}\frac{\partial F\left(x_{(i)}|b,\delta,\alpha,\theta\right)}{\partial\theta}=0$$

3.6.2 Methods of Anderson-Darling and Right-tail Anderson-Darling

Another type of minimum distance estimators is based on Anderson-Darling statistic and is known as the Anderson-Darling estimator (ADE). The Anderson-Darling test is like Cramér-von-Mises criterion except that the integral is of a weighted version of the squared difference, where the weighting relates the variance of the empirical distribution function. The Anderson-Darling test was developed T.W. Anderson and D.A. Darling (T. W. Anderson, D. A. Darling, (1952). Anderson, T., & Darling, D. (1954).) as an alternative to other statistical tests for detecting sample distributions departure from normality. The Anderson-Darling estimates of the parameters are obtained by minimizing, with respect to b,δ,α,θ, the function:

$$A(b,\delta,\alpha,\theta)=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[logF\left(x_{(i)}|b,\delta,\alpha,\theta\right)+\log\bar{F}\left(x_{(n+1-i)}|b,\delta,\alpha,\theta\right)\right] \quad (26)$$

$$\sum_{i=1}^{n}(2i-1)\left[\frac{F'_b\left(x_{(i)}|b,\delta,\alpha,\theta\right)}{F\left(x_{(i)}|b,\delta,\alpha,\theta\right)}-\frac{\bar{F}'_b\left(x_{(n+1-i)}|b,\delta,\alpha,\theta\right)}{\bar{F}\left(x_{(n+1-i)}|b,\delta,\alpha,\theta\right)}\right]=0$$

$$\sum_{i=1}^{n}(2i-1)\left[\frac{F'_\delta\left(x_{(i)}|b,\delta,\alpha,\theta\right)}{F\left(x_{(i)}|b,\delta,\alpha,\theta\right)}-\frac{\bar{F}'_\delta\left(x_{(n+1-i)}|b,\delta,\alpha,\theta\right)}{\bar{F}\left(x_{(n+1-i)}|b,\delta,\alpha,\theta\right)}\right]=0$$

$$\sum_{i=1}^{n}(2i-1)\left[\frac{F'_\alpha\left(x_{(i)}|b,\delta,\alpha,\theta\right)}{F\left(x_{(i)}|b,\delta,\alpha,\theta\right)}-\frac{\bar{F}'_\alpha\left(x_{(n+1-i)}|b,\delta,\alpha,\theta\right)}{\bar{F}\left(x_{(n+1-i)}|b,\delta,\alpha,\theta\right)}\right]=0$$

$$\sum_{i=1}^{n}(2i-1)\left[\frac{F'_\theta\left(x_{(i)}|b,\delta,\alpha,\theta\right)}{F\left(x_{(i)}|b,\delta,\alpha,\theta\right)}-\frac{\bar{F}'_\theta\left(x_{(n+1-i)}|b,\delta,\alpha,\theta\right)}{\bar{F}\left(x_{(n+1-i)}|b,\delta,\alpha,\theta\right)}\right]=0$$

The Right-tail Anderson-Darling estimates of the parameters are obtained by minimizing, with respect to b,δ,α,θ, the function:

$R(b,\delta,\alpha,\theta)=\frac{n}{2}-2\sum_{i=1}^{n}F\left(x_{(i)}\middle|b,\delta,\alpha,\theta\right)-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\bar{F}\left(x_{(n+1-i)}|b,\delta,\alpha,\theta\right)$ (27)

These estimates can also be obtained by solving the non-linear equations:

$$-2\sum_{i=1}^{n}\frac{F_b'\left(x_{(i)}|b,\delta,\alpha,\theta\right)}{F\left(x_{(i)}|b,\delta,\alpha,\theta\right)}+\frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\bar{F}_b'\left(x_{(n+1-i)}|b,\delta,\alpha,\theta\right)}{\bar{F}\left(x_{(n+1-i)}|b,\delta,\alpha,\theta\right)}=0$$

$$-2\sum_{i=1}^{n}\frac{F_\delta'\left(x_{(i)}|b,\delta,\alpha,\theta\right)}{F\left(x_{(i)}|b,\delta,\alpha,\theta\right)}+\frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\bar{F}_\delta'\left(x_{(n+1-i)}|b,\delta,\alpha,\theta\right)}{\bar{F}\left(x_{(n+1-i)}|b,\delta,\alpha,\theta\right)}=0$$

$$-2\sum_{i=1}^{n}\frac{F_\alpha'\left(x_{(i)}|b,\delta,\alpha,\theta\right)}{F\left(x_{(i)}|b,\delta,\alpha,\theta\right)}+\frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\bar{F}_\alpha'\left(x_{(n+1-i)}|b,\delta,\alpha,\theta\right)}{\bar{F}\left(x_{(n+1-i)}|b,\delta,\alpha,\theta\right)}=0$$

$$-2\sum_{i=1}^{n}\frac{F_\theta'\left(x_{(i)}|b,\delta,\alpha,\theta\right)}{F\left(x_{(i)}|b,\delta,\alpha,\theta\right)}+\frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\bar{F}_\theta'\left(x_{(n+1-i)}|b,\delta,\alpha,\theta\right)}{\bar{F}\left(x_{(n+1-i)}|b,\delta,\alpha,\theta\right)}=0$$

4. Application

4.1 Simulation Study

Within this part, we conduct Monte Carlo simulation research to assess the effectiveness of several estimate approaches in determining the parameters of the MO-TLHL-BX distribution. We evaluate the suggested estimators by using the Kolmogorov-Smirnov test.

This technique relies on the KS statistic: $D_n=\max_x|F_n(x)-F(x|b,\delta,\alpha,\theta)|$

where $\max_x$ denotes the maximum of the set of distances, $F_n(x)$ is the empirical distribution function, and $F(x|b,\delta,\alpha,\theta)$ is the cumulative distribution function.

Firstly, we provided an algorithm to generate a random sample from the MO-TLHL-BX distribution for given values of its parameters and sample size n. The following procedure was adopted:

1. Set n, $\Theta=(b,\delta,\alpha,\theta)$ and initial value x^0.
2. Generate $U\sim Uniform(0,1)$.
3. Update x^0 by using the Newton's formula.

$$x^*=x^0-R(x^0,\Theta)$$

where, $R(x_0, \Theta) = \frac{F_X(x^0,\Theta)-U}{f_X(x^0,\Theta)}$, $F_X(\cdot)$ and $f_X(\cdot)$ are cdf and pdf of the MO-TLHL-BX distribution, respectively.

4. If $|x^0 - x^*| \leq \epsilon$ (very small, $\epsilon > 0$ tolerance limit), then store $x = x^*$ as a sample from MO-TLHL-BX distribution.
5. If $|x^0 - x^*| > \epsilon$, then set $x^0 = x^*$ and go to step 3.
6. Repeat steps 3-5, n times for $x_1, x_2, \dots, x_n$ respectively.

For this purpose, we take $b = 0.5, \delta = 0.5, \alpha = 0.5, \theta = 1.9$ arbitrarily and $n = 10, 20, \dots, 100$. All the algorithms are coded in R, a statistical computing environment and we used algorithm given above for simulations purpose.

From the results of the simulation study, it is observed that the method of Maximum Likelihood Estimation (MLE) is the most efficient method compared to others for estimating the parameters of the MO-TLHL-BX distribution since it has the minimum value of Kolmogorov-Smirnov test (Table 1).

Table 1. The methods of estimation and their respective Kolmogorov-Smirnov test value.

i	Methods of Estimations	K-S test	Ranking
1	Maximum Product Spacing Estimating	0.0375580	4
2	Least Square Estimation	0.0366624	3
3	Weighted Least Square Estimation	0.0342950	2
4	L-Moment Estimation	0.0386830	5
5	**Maximum Likelihood Estimation**	**0.0333950**	**1**
6	Maximum Product Spacing Estimating	0.0403951	6
7	Anderson-Darling Estimation	0.0408900	7
8	Right-tail Anderson-Darling	0.0413950	8

4.2 Real Data Set

We will assess the effectiveness of the expanded distribution in this section. This research employs an authentic data set to illustrate our model's superior performance when compared to alternative models that were implemented on the identical data set. The information provided concerns the daily influx of new COVID-19 cases in Albania between July 1st and August 1st, 2022, with respect to the advent of a novel strain of the virus.

The data is collected from the official site of the [Albania COVID-Coronavirus Statistics-Worldometer (worldometers.info)].

The data are as follows: 619; 671; 549; 147; 974; 945; 973; 1,001; 962; 290; 561; 1,215; 1,563; 1,502; 1,461; 1,326; 1,062; 427; 1,846; 1,480; 1,336; 1,373; 1,158; 965; 233; 1,666; 1,261; 1,228; 1,084; 1,019; 716.

Table 2. MLEs and comparison criteria for the COVID-19 case fatality ratio in Albania.

Distribution	Parameter Estimate	$-\ell$	AIC	BIC	CAIC
MO-TLHL-BX $(b, \delta, \alpha, \theta)$	0.585, 0.025, 0.126 0.018	89.91528	142.1146	139.631	147.164
Beta Burr Type X $(\alpha, \lambda, \mu, \delta)$	0.141, 4.384, 1.081 0.112	131.04398	175.1326	170.365	184.574
Generalized Burr type X (α, λ, ρ)	0.028, 0.115, 0241	194.14108	267.2326	268.951	278.247
Exponentiated Burr type X (θ, β)	0.041, 0.365	200.11628	317.4656	321.614	329.543

To assess the distribution models, several metrics such as AIC (Akaike information criterion), CAIC (corrected Akaike information criterion), and BIC are considered for the given dataset. A more optimal distribution is characterized by lower values of the criterion.

$$AIC = -2log\ell(\tilde{x}, \alpha, \beta, \theta) + 2p$$

$$CAIC = AIC + \frac{2p(p+1)}{n-p-1}$$

$$BIC = -2log\ell(\tilde{x}, \alpha, \beta, \theta) + plog(n)$$

The p-value indicates the number of parameters that will be estimated from the data, whereas n represents the sample size.

According to the findings shown in Table 2, our analysis demonstrates that the Marshall-Olkin Topp-Leone Half-Logistic-Burr type X distribution has superior goodness of fit compared to other models, namely the Burr-Type X, Beta Burr Type X, Generalized Burr type X, Exponentiated Burr type X.

Conclusion

Utilizing the Marshall-Olkin Topp-Leone Half-Logistic-G family distributions, this research paper introduces the Marshall-Olkin Topp-Leone Half-Logistic-Burr type X, an innovative distribution. An examination of numerous statistical characteristics of the distribution was conducted, and an endeavour was made to construct a model that could estimate its parameters. A comparative effectiveness analysis of multiple estimators was performed through simulation research employing the Kolmogorov-Smirnov test. The current investigation examines a genuine set of COVID-19 data to showcase the versatility of the proposed model in contrast to the degree of precision attained by alternative distributions. It is hypothesized that the application of this broadened distribution holds promises for investigation in additional fields of study.

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