ON THE $(\alpha, \beta, \varphi, \delta)$ – CONTRACTIONS OF WEAKLY-COMPATIBLE ORBITAL-CYCLIC MAPPINGS IN EXTENDED CONE METRIC SPACES

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Abstract

In this paper we give a fixed point result provided by the type $(\alpha, \beta, \varphi, \delta)$-contraction for weakly-compatible orbital-cyclic mappings on extended cone metric spaces where φ is a comparison function and δ is a non-negative real number. We describe the iterative process finalizing the convergence result in such maps. The study focuses on a class of mappings known as weakly-compatible orbital-cyclic mappings. These mappings have specific properties that are investigated in this paper. The purpose of this paper is to establish conditions under which such mappings guarantee the existence of fixed points in extended cone metric spaces.

***Key words*:** (α, β)*-orbital-cyclic admissible quartet, Weakly compatible, Extended cone metric space, Banach operator pair.*

Përmbledhje

Në këtë punim ne japim një rezultat për pikën fikse të përftuar nga kontraksioni i tipit $(\alpha, \beta, \varphi, \delta)$ për pasqyrimet orbital-ciklikë dobët të përputhshëm në hapësirat kon-metrike të zgjeruara, ku φ është një funksion krahasimi dhe δ është një numër real jonegativ. Ne përshkruajmë procesin përsëritës (iteracionet) që finalizon rezultatin e konvergjencës të kësaj kategorie pasqyrimesh. Studimi fokusohet në një klasë pasqyrimesh të njohura si pasqyrimet orbital-ciklike dobët të pëputhshme. Këto pasqyrime kanë veti specifike që i gjeni të hulumtuara në këtë punim. Qëllimi i këtij punimi është të vendosë kushtet në të cilat pasqyrime të tilla garantojnë ekzistencën e pikave fikse në hapësirat e zgjeruara kon metrike.

Fjalë kyçe: *Katërshja* (α, β)*-orbital-ciklike e pranueshme, Dobët i përputhshëm, Hapësirat e zgjeruara kon-metrike, Çift operatorësh Banach.*

1. Introduction and Preliminaries

Research in fixed point theory may involve the development of new concepts, theorems, and techniques to analyze different type of metric spaces and to find fixed points for various classes of functions. In this way, these are many generalizations of fixed point theory on such spaces which offer interest for us, starting from the cone metric spaces, extended to cone *b*-metric spaces, later to cone *b*-quasi metric spaces and to extended cone *b*-quasi metric spaces. In his paper, Haokip et al. gave a common fixed point result for a class of contractive mappings which is a generalization of the Alqahtani contractive condition. We propose a new contraction of type $(\alpha,\beta,\varphi,\delta)$ for two pairs of (α,β)-orbital-cyclic admissible and simultaneously weakly-compatible functions, which generalize the Haokip et al. result.

Some useful definitions which be used in function of our main result are introduced and reproduced below.

One of the most interesting generalizations of metric spaces was introduced in 2007 by Huang and Zhang.

Let E be a real Banach space and P be a subset of E.

Definition 1.1. (Huang & Zhang, 2007) Let P be a nonempty subset of E, where E is an ordered Banach space. The set P is called *cone* if and only if:

(i) P is closed, nonempty and $P \neq \{\theta\}$,

(ii) $a,b \in \mathbb{R}$, $a,b \geq 0$, $x,y \in P$ implies $ax+by \in P$,

(iii) $x \in P$ and $-x \in P$ implies $x=\theta$.

The cone P is called *normal* if, there is a positive real number K such that, for all x,y in P we have:

$$0 \leq x \leq y \Rightarrow \|x\| \leq K\|y\| \tag{2.1}$$

The positive number K is called *the normality constant* of P.

Definition 1.2. (Huang & Zhang, 2007) Let P be a cone and X a non-empty set. The function $d: X \times X \to P$ is called a *cone metric* if it satisfies the following conditions:

(*c1*) $d(x,y) \in P$ that is $0 \leq d(x,y)$ for and $d(x,y)=0$ iff $x=y$,

(*c2*) $d(x,y)=d(y,x)$ for all $x,y \in X$,

$(c3)$ $d(x,y) \le d(x,z) + d(z,y)$ for all $x,y,z \in X$.

The pair (X,d) is called a *cone metric space*.

The concept of cone metric space and fixed point theory on these spaces has been developed from many authors in their works. One of the most important extension of cone metric spaces is established by the following concept introduced by Czerwik.

Definition 1.3. (Czerwik, 1998) Let be X a nonempty set and $s \ge 1$be a given real number. A function $d : X \times X \to \mathbb{R}^+$is called a *b*-metric if, for all $x,y,z \in X$ it satisfies the conditions:

$(b1)$ $d(x,y) = 0$ iff $x = y$,

$(b2)$ $d(x,y) = d(y,x)$,

$(b3)$ $d(x,z) \le s\left[d(x,y) + d(y,z)\right]$.

The pair (X,d) is called *b*-metric space with parameter *s*.

Kamran et al. in 2017, introduced a new type of generalized metric space by taking a two-variables function $\theta(x,y)$ instead of the parameter *s*.

Definition 1.4. (Kamran, Samreen, Ain, 2017) Let X be a nonempty set and $\theta : X \times X \to [1,+\infty)$. A function $d_\theta : X \times X \to [0,+\infty)$ is an extended *b*-metric, if for all $x,y,z \in X$ it satisfies:

$(d_\theta 1)$ $d_\theta(x,y) = 0$ iff $x = y$,

$(d_\theta 2)$ $d_\theta(x,y) = d_\theta(y,x)$, for all $x, y \in X$,

$(d_\theta 3)$ $d_\theta(x,z) \le \theta(x,z)\left(d_\theta(x,y) + d_\theta(y,z)\right)$ for all $x,y,z \in X$.

The pair (X,d_θ) is called an extended *b*-metric space.

Example1.5. (Kamran, Samreen, Ain, 2017)

Let $X = \left(C[a,b],\mathbb{R}\right)$ be the space of all continuous real valued functions defined on $[a,b]$. Note that X is complete extended *b*-metric space by considering $d_\theta(x,y) = \sup_{t\in[a,b]} \left|x(t) - y(t)\right|^2$, with $\theta(x,y) = |x(t)| + |y(t)| + 2$, where $\theta : X \times X \to [1,+\infty)$.

Example 1.6. Let $X=\{1,\ 2,\ 3\}$ and $\theta : X \times X \to [1,+\infty)$ defined by $\theta(x,y)=1+xy.$ Let be the cone $P=\{(a,b)\in \mathbb{R}^2 : a,b\geq 0\}.$ Define the function d_θ as follows:

$$d_\theta(1,2)=d_\theta(2,1)=(10,10)$$
$$d_\theta(1,3)=d_\theta(3,1)=(20,20)$$
$$d_\theta(2,3)=d_\theta(3,2)=(30,30)$$
$$d_\theta(1,1)=d_\theta(2,2)=d_\theta(3,3)=(0,0)=\theta$$

It is clear by the definition of d_θ that the first and second conditions of extended cone metric space are fulfilled. Let now check the third condition:

$d_\theta(x,z)\leq \theta(x,z)\left[d_\theta(x,y)+d_\theta(y,z)\right]$ for all $x,y,z\in X$ or equivalently,

$$\theta(x,z)\left[d_\theta(x,y)+d_\theta(y,z)\right]-d_\theta(x,z)\geq 0.$$

$$\theta(1,3)\left[d_\theta(1,2)+d_\theta(2,3)\right]-d_\theta(1,3)=4\left[(10,10)+(30,30)\right]-(20,20)=(140,140)\in P$$
$$\theta(1,2)\left[d_\theta(1,3)+d_\theta(3,2)\right]-d_\theta(1,2)=3\left[(20,20)+(30,30)\right]-(10,10)=(140,140)\in P$$
$$\theta(2,3)\left[d_\theta(2,1)+d_\theta(1,3)\right]-d_\theta(2,3)=7\left[(10,10)+(20,20)\right]-(30,30)=(180,180)\in P$$

Then (X,d_θ) is an extended cone metric space.

Recently giving results in fixed point theory for cone metric spaces, Das and Bag in 2022 introduced the concept of extended cone metric space using this time a three-variables function $\theta(x,y,z).$

Definition 1.7. (Das & Bag, 2022) Let be X a nonempty set and $\theta : X\times X\times X \to [1+\infty).$ Let $d_\theta : X\times X \to \mathbb{R}^+$ be a function which satisfies the following conditions:

$(d_\theta 1)$ $d_\theta(x,y)\geq 0$ and $d_\theta(x,y)=0$ iff $x=y,$

$(d_\theta 2)$ $d_\theta(x,y)=d_\theta(y,x),$ for all $x,y\in X,$

$(d_\theta 3)$ $d_\theta(x,z)\leq \theta(x,y,z)\left(d_\theta(x,y)+d_\theta(y,z)\right)$ for all $x,y,z\in X.$

The function d_θ is called extended cone metric on X and the pair (X,d_θ) is called extended cone metric space.

Some basic topological notions such as convergence, Cauchy sequences, continuity and completeness on extended cone metric spaces are defined as follows:

Definition 1.8. Consider a sequence $\{x_n\}$ in an extended cone metric space (*X*, *d*) and *P* be a normal cone in *E* with normality constant *K*.

Then

(i) $\{x_n\}$ converges to *x* if for every $c \in E$ with *c* > 0, $\exists N \in \mathbb{N}$, such that for all $n \geq N,\ d(x_n, x) < c.$ Denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \xrightarrow[n\to\infty]{} x.$

(ii) $\{x_n\}$ is said to be Cauchy in *X* if for every $c \in E$ with *c* >0, $\exists N \in \mathbb{N}$, such that for all $n,\ m \geq N, d(x_n, x_m) < c.$

(iii) the mapping $T : X \to X$ is said to be continuous at a point $x \in X$ if for every sequence $\{x_n\}$ converging to *x*, it follows that $\lim_{n\to\infty} Tx_n = T \lim_{n\to\infty} x_n = Tx.$

(iv) (*X*, *d*) is said to be a complete cone metric space if every Cauchy sequence in *X* is convergent in *X*.

Alqahtani et al. in their work gave the notion of (α, β)-*orbital - cyclic admissible* pair and proved some fixed point results for a couple of orbital cyclic functions in extended *b*-metric spaces.

Definition 1.9. (Alqahtani, Fulga & Karapınar, 2018) Let *S*, *T* are two self-mappings on a complete extended cone metric space (X, d_θ). Suppose that there are two functions $\alpha, \beta : X \times X \to [0, \infty)$ such that for any $x \in X$,

$$\begin{aligned} &\alpha(x, Tx) \geq 1 \quad \Rightarrow \quad \beta(Tx, STx) \geq 1 \quad \text{and} \\ &\beta(x, Sx) \geq 1 \quad \Rightarrow \quad \alpha(Sx, TSx) \geq 1 \end{aligned} \tag{2.2}$$

Then (*S*, *T*) is called (α, β) -*orbital -cyclic admissible* pair.

Based on this definition, Haokip et al. in 2022, introduced the following

Definition 2.0. (Haokip, Goswami, Tripathy, 2022) Let *S*, *T* are two self-mappings on a complete extended cone metric space (X, d_θ) with f injective and $\alpha, \beta : X \times X \to [0, \infty)$ be two mappings such that for any $x \in X$

$$\begin{aligned} &\alpha(fx, fTx) \geq 1 \quad \Rightarrow \quad \beta(fTx, fSTx) \geq 1 \quad \text{and} \\ &\beta(fx, fSx) \geq 1 \quad \Rightarrow \quad \alpha(fSx, fTSx) \geq 1 \end{aligned} \tag{2.3}$$

Then, f, S, T are said to be an (α, β) -*orbital -cyclic admissible triplet*.

The *Banach operator* pair notion has been used in Haokip et al. paper. It was first introduced by Subrahmanyam, extended by Chen and Li and later, by Öztürk and Başarir.

Definition 2.1. (Öztürk & Başarir, 2011) Let f and T are self-mappings on an extended cone metric space (X, d_θ). Then the pair (f, T) is said to be a *Banach operator* pair if, for some $k > 0$,

$$d_\theta(fTx, Tx) \le k d_\theta(Tx, x) \qquad \text{for all } x \in X.$$

Definition 2.2. (Haokip, Goswami, Tripathy, 2022) Let f and T are self-mappings on an extended cone metric space (X, d_θ). T is said to be *Cauchy-commutative* with respect to f if for any sequence $\{x_n\}$ in X such that $\{x_n\}$ is a Cauchy sequence, $fTx = Tfx$ for each x in $\{x_n\}$.

In another generalization, introduced in 1984 by M.S. Khan, M. Swalech and S. Sessa, they use a control function which they called an *altering distance function.*

This generalization is extended by Haokip, Goswami, Tripathy as follows:

Definition 2.3. (Haokip, Goswami, Tripathy, 2022) A function $\varphi : [0,\infty) \to [0,\infty)$ is called a *sub-additive altering distance function if,*

(i) $\varphi(x+y) \le \varphi(x) + \varphi(y) \qquad \forall x, y \in [0,\infty)$

(ii) φ is an *altering distance function* (Faraji, Nourouzi, 2017) (i.e., φ is continuous, strictly increasing and $\varphi(t) = 0$ if and only if $t = 0$)

Haokip et al. concluded in their research, the main result of which is represented as well from following lemma and theorem.

Lemma 2.4 (Haokip, Goswami, Tripathy, 2022) *Let* (X, d_θ) *be an extended b-metric space. If there exists* $q \in [0,1)$ *such that the sequence* $\{x_n\}$ *for an arbitrary* $x_0 \in X$ *satisfies*

$$\lim_{n,m\to\infty} \theta(x_n, x_m) < \frac{1}{q} \text{ and also } \varphi\big(d_\theta(x_n, x_{n+1})\big) \le q \cdot \varphi\big(d_\theta(x_{n-1}, x_n)\big) \quad (2.4)$$

for all positive integer n, then the sequence $\{x_n\}$ *is Cauchy in X.*

Theorem 2.5 (Haokip, Goswami, Tripathy, 2022) Let (X, d_θ) be *complete extended b-metric space and* $f, S, T : X \to X$ *to be an* (α, β) *-orbital-cyclic* *admissible triplet of mappings on X. Let* (f,S) *and* (f,T) *be two Banach operator pairs such that for all* $x, y \in X$

$$\begin{aligned}&\alpha(fx, fTx)\beta(fy, fSy)d_\theta(fTx, fSy)\\&\le k_1\varphi(d_\theta(fx, fy))+k_2\varphi(d_\theta(fx, fTx))+k_3\varphi(d_\theta(fy, fSy))\end{aligned} \tag{2.5}$$

for some $k_1, k_2 \ge 0, k_3 > 0$ *and* $k_1+k_2+k_3<1.$

Suppose that there exists $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \ge 1.$ *Let for each* $x_0 \in X$, $\lim_{n.m\to\infty}\theta(fx_n, fx_m) < \dfrac{1-k_3}{k_1+k_2}$, *where* $x_{2n} = Sx_{2n-1}$ *and* $x_{2n-1} = Tx_{2n-2}$ *for each* $n \in \mathbb{N}$. *Then*

(i) f, S *and* T *have a unique common fixed point, if S and T are continuous and Cauchy commutative with respect to f and* $\alpha(fz, fz) \ge 1$ *for* $z \in CF(S,T)$, *where* $CF(S,T)$ *is the set of common fixed points of mappings S and T.*

(ii) f, S *and* T *have a unique common fixed point, if f is continuous and if* $\{x_n\} \subseteq X$ *is a sequence such that* $\lim_{n\to\infty} x_n = z$, *then* $\alpha(fz, fTz) \ge 1$ *and* $\beta(fz, fSz) \ge 1.$

Main result

We prove a fixed point theorem for two couples of (α,β)*-orbital-cyclic admissible* and *weakly-compatible* functions in extended cone metric space equipped with θ three-variables function. The contraction $(\alpha,\beta,\varphi,\delta)$ proposed by us extend (α,β)-orbital-cyclic admissible properties by using the continuous map φ and a nonnegative constant δ.

First, we give this definition which extend the notion of (α,β)*-orbital -cyclic admissible quartet*.

Definition 2.6. Let *S, T* are two self-mappings on a complete extended cone metric space (X, d_θ) with *f, g* injective and $\alpha, \beta : X \times X \to [0,\infty)$ be two mappings such that for any $x \in X$

$$\begin{aligned}&\alpha(fx, gTx) \ge 1 \;\Rightarrow\; \beta(fTx, gSTx) \ge 1 \quad \text{and}\\&\beta(fx, gSx) \ge 1 \;\Rightarrow\; \alpha(fSx, gTSx) \ge 1.\end{aligned} \tag{2.6}$$

Then, f, g, S and T *are said to be an* (α, β) *-orbital-cyclic admissible quartet.*

The following Lemma is important for the proof of theorem which we give in this paper.

Lemma 2.7. *Let* (X, d_θ) *be an extended cone metric space. If there exists* $q \in [0,1)$ *such that the sequence* $\{x_n\}$ *for an arbitrary* $x_0 \in X$ *satisfies*

$$\lim_{n,m\to\infty} \theta(x_n, x_m, x_{n+1}) < \frac{1}{q} \quad \textit{and}$$

$$d_\theta(x_{n+1}, x_n) \leq q \cdot d_\theta(x_n, x_{n-1}) \tag{2.7}$$

for any $n \in \mathbb{N}$, *then the sequence* $\{x_n\}$ *is Cauchy in X.*

Theorem 2.8. Let (X, d_θ) *be a complete extended cone metric space and* $f, g, S, T : X \to X$ *to be an* (α, β) *-orbital-cyclic admissible two pairs* (f, S) *and* (g, T) *on X. Let* (f, S) *and* (g, T) *be Banach operator pairs and weakly-compatible such that for all x, y* $\in X$

$$\begin{aligned} &\alpha(fx, gTx)\beta(fy, gSy)d_\theta(fTx, gSy) \\ &\leq \varphi\left(\max\left\{k_1 d_\theta(fx, gTx), k_2(d_\theta(fy, gSy), k_3 d_\theta(fTx, gSy)\right\}\right) \\ &+\delta \min\left\{d_\theta(fx, fTx), d_\theta(gy, gSy), d_\theta(fTx, gSy), d_\theta(fx, gSy)\right\}. \end{aligned} \tag{2.8}$$

for some $k_1, k_2 \geq 0, k_3 > 0$ *and* $k_1 + k_2 + k_3 < 1.$

Suppose that, there exists $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1.$ *Let for all* $x_0 \in X$ $\lim\limits_{n.m\to\infty} \theta(fx_n, fx_m, x_{n+1}) < \dfrac{1-k_3}{k_1+k_2}$, *where* $x_{2n} = Sx_{2n-1}$ *and* $x_{2n-1} = Tx_{2n-2}$ *for each* $n \in \mathbb{N}$. *Then f, g, S, T have a unique common fixed point if S, T are continuous and Cauchy commutative with respect to f and g and* $\alpha(fz, gz) \geq 1$ *for z* $\in CF(S,T)$ *where* $CF(S,T)$ *is denoted the set of common fixed points of mappings S and T.*

Proof. By assumption in the conditions of theorem, there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1.$ Let be $x_1 = Tx_0$ and $x_2 = Sx_1$. Continuing inductively, we construct the sequence $\{x_n\}$ taking

$$x_{2n} = Sx_{2n-1} \textit{ and } x_{2n-1} = Tx_{2n-2} \textit{ for each } n \in \mathbb{N}. \tag{2.9}$$

Since $\alpha(x_0, Tx_0) \geq 1.$ using the fact that (f, g, S, T) is an (α, β) -orbital-cyclic admissible quartet, we can derive as in Alqahtani et al. paper, obtaining

$$\alpha(fx_{2n}, gx_{2n+1}) \geq 1 \quad \text{and} \quad \beta(fx_{2n+1}, gx_{2n+2}) \geq 1 \text{ for all } n \in \mathbb{N} \qquad (2.10)$$

For all nonnegative integers n we can assume, without loss generality, that $x_n \neq x_{n+1}$. This holds because, if there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then by the construction of the sequence $\{x_n\}$, we can show that is a common fixed point of (f, g, S, T) and the proof is done.

To check this, let consider the following two cases for n_0.

(i) If n_0 is even, thus $n_0 = 2m$ then $x_{2m} = x_{2m+1} = Tx_{2m}$ and x_{2m} is a fixed point of T and S. Since (f, T) and (g, S) are Banach operator pairs, we have for some

$$d_\theta(fx_{2m}, x_{2m}) = d_\theta(fTx_{2m}, Tx_{2m}) \leq kd_\theta(Tx_{2m}, x_{2m}) = 0 \quad \text{and}$$
$$d_\theta(gx_{2m}, x_{2m}) = d_\theta(gSx_{2m}, gSx_{2m}) \leq kd_\theta(Sx_{2m}, x_{2m}) = 0$$

showing that x_{2m} is also fixed point of f and g.

We also claim that $x_{2m} = x_{2m+1} = Sx_{2m+1} = Tx_{2m}$. Suppose on the contrary that $Sx_{2m+1} \neq Tx_{2m}$. Then, replacing $x = x_{2m}$, $y = x_{2m+1}$ in (1.8) and using (1.9), (1.10) we have

$$\varphi\left(d_\theta(fTx_{2m}, gSx_{2m+1})\right)$$
$$\leq \varphi\left(\max\left\{k_1 d_\theta(fx_{2m}, gx_{2m+1}), k_2(d_\theta(fx_{2m}, fTx_{2m}), k_3 d_\theta(gx_{2m+1}, gSx_{2m+1})\right\}\right)$$
$$+\delta \min\left\{d_\theta(fx_{2m}, fTx_{2m}), d_\theta(gx_{2m+1}, gSx_{2m+1}), d_\theta(fTx_{2m}, gx_{2m+1}), d_\theta(fx_{2m}, gSy_{2m+1})\right\}.$$

This inequality can simplify using the injectivity of the functions f, g and the above results, as

$$\varphi\left(d_\theta(fTx_{2m}, gSx_{2m+1})\right) \leq k_3 \varphi\left(d_\theta(fTx_{2m}, gSx_{2m+1})\right).$$

Now, we can use the function φ property, as a subadditive altering distance and the fact that $k_3 < 1$ obtaining $d_\theta(fTx_{2m}, gSx_{2m+1}) < d_\theta(fTx_{2m}, gSx_{2m+1})$ which is a contradiction.

Hence, $d_\theta(Tx_{2m}, Sx_{2m+1}) = 0$ and $x_{2m} = x_{2m+1} = Sx_{2m+1} = Tx_{2m}$ which implies that $u = x_{2m} = x_{2m+1}$ is a fixed point for all f, S, g, T.

(ii) If n_0 is odd, similarly, following an analogue procedure we can obtain the same result.

Hence, we can continue the proof assuming that for all nonnegative integers $x_n \neq x_{n+1}$. Now, we must show that $f(x_n)$ and $g(x_n)$ are Cauchy sequences. In order to show that our sequences fulfill the conditions of Lemma (2.7), it would be enough to study the cases when $x = x_{2n}, y = x_{2n+1}$ and $x = x_{2n}, y = x_{2n-1}$.

Case (a) Let $x = x_{2n}$ and $y = x_{2n+1}$. Using (1.8) and (1.10) we derive that

$$\begin{aligned} 0 &< d_\theta(fx_{2n+1}, fx_{2n+2}) \\ &= d_\theta(fTx_{2n}, fSx_{2n+1}) \\ &\leq q d_\theta(fx_{2n}, fx_{2n+1}) \end{aligned}$$

Under the monotony of φ from the last inequality we obtain

$$\begin{aligned} \varphi\left(d_\theta(fx_{2n+1}, fx_{2n+2})\right) &= \varphi\left(d_\theta(fTx_{2n}, fSx_{2n+1})\right) \\ &\leq q \cdot \varphi\left(d_\theta(fx_{2n}, fx_{2n+1})\right) \end{aligned} \tag{2.11}$$

where $q = \dfrac{k_1 + k_2}{1 - k_3} < 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Case (b) Let $x = x_{2n}$ and $y = x_{2n-1}$. Similarly, we obtain

$$\varphi\left(d_\theta(fx_{2n}, fx_{2n+1})\right) \leq q \cdot \varphi\left(d_\theta(fx_{2n-1}, fx_{2n})\right) \tag{2.12}$$

Analogously, we can write that

$$\varphi\left(d_\theta(gx_{2n}, gx_{2n+1})\right) \leq q \cdot \varphi\left(d_\theta(gx_{2n-1}, gx_{2n})\right) \tag{2.13}$$

By using Lemma (2.7) it follows that $f(x_n)$ and $g(x_n)$ are Cauchy sequences. From the completeness of (X, d_θ) these sequences are both convergent on X. Using the fact that $(f,T), (g,S)$ *a*re Banach operators and (S,T) are weakly compatible there exists $u \in X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ and consequently

$$\begin{aligned} \lim_{n\to\infty} fx_{2n} &= \lim_{n\to\infty} fx_{2n+1} = u \\ \lim_{n\to\infty} gx_{2n} &= \lim_{n\to\infty} gx_{2n+1} = u. \end{aligned}$$

(a) Since (S,T) are weakly compatible and continuous, we have

$$Su = S\left(\lim_{n\to\infty} fx_{2n-1}\right) = \lim_{n\to\infty} Sfx_{2n-1} = \lim_{n\to\infty} fSx_{2n-1} = \lim_{n\to\infty} fx_{2n} = u$$
$$Tu = T\left(\lim_{n\to\infty} fx_{2n}\right) = \lim_{n\to\infty} Tfx_{2n} = \lim_{n\to\infty} fTx_{2n} = \lim_{n\to\infty} fx_{2n-1} = u$$

Thus, *u* is common fixed point of *S* and *T*.

If, for assumption u' is another common fixed point of *S* and *T*, then $u, u' \in CF(S,T)$. Consequently, $\alpha(fu, gu) \geq 1$ and $\alpha(fu', gu') \geq 1$. Then, $\alpha(fu, gTu) \geq 1$ and $\beta(fu', gSu') \geq 1$(from the definition 2.6, $\alpha(fx, gTx) \geq 1$ implies $\beta(fTx, gSTx) \geq 1$). Now, using (1.8) where $x = u,$, $y = u'$ we have

$$\alpha(fu, gTu)\beta(fu', gSu')d_\theta(fTu, gSu')$$
$$\leq \varphi\left(\max\left\{k_1 d_\theta(fu, gTu), k_2(d_\theta(fu', gSu'), k_3 d_\theta(fTu, gSu')\right\}\right)$$
$$+\delta \min\left\{d_\theta(fu, fTu), d_\theta(gu', gSu'), d_\theta(fTu, gSu'), d_\theta(fu, gSu')\right\}.$$

or simply $d_\theta(fu, gu') \leq k_3 d_\theta(fu, gu')$ *which is possible only when* $fu = gu'$.

Changing the place of *x* with *y* reusing (1.8) similarly we obtain $d_\theta(gu, fu') \leq k_3 d_\theta(gu, fu')$ which is possible only when $gu = fu'$.

So, $fu = gu = fu' = gu'$ and from the injectivity of *f* and *g* it follows that $u = u'$. Thus, the common fixed point of *S* and *T* is unique.

Since (f,S) and (g,T) are weakly-compatible we have $fSu = Sfu$ and $gTu = Tgu$ for $u \in F(S,T)$ from which it follows that *fu* and *gu* are two other fixed points of *S* and *T*. From the uniqueness of such fixed points, it follows that *f*, *g*, *S*, *T* have the common fixed point *u* which is unique.

Remark 2. 9. Theorem 2.8 generalize the theorem 2.5 (Haokip). Indeed, if we take $g = f$ in Theorem 2.8 the result is clear.

We can show that Alqahtani et al. theorem can be interpreted as corollary of Theorem 2.8.

Corollary 2.10. *Let T, S be two self-mappings on a complete extended cone metric space* (X, d_θ) *such that the pair (T, S) is* (α, β)*-orbital-cyclic admissible. Suppose that there is a constant δ > 0 and a continuous comparison function* $\varphi : X \to E$ *such that* $\lim_{n\to\infty} \left\|\varphi^n(t)\right\| = 0$ *and*

(i) *for each* $x_0 \in X$, $\lim_{n,m\to\infty}\theta(x_n, x_m, x_{n+1}) < \frac{1-k}{k}$,

where $x_{2n} = Sx_{2n-1}$ *and* $x_{2n+1} = Tx_{2n}$ *for each* $n \in \mathbb{N}$,

(ii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$,

(iii) *S and T are continuous and satisfies the following inequality*

$$\alpha(x,Tx)\beta(y,Sy)d_\theta(Tx,Sy) \leq k\left(d_\theta(x,Tx) + d_\theta(y,Sy)\right)$$

Then the pair (*T*, *S*) has a common fixed point u, that is $Tu = u = Su$.

Proof. If we take in theorem 2.8 $k_1 = 0, k_2 = k_3 = k$ $fx = x$, $gy = y$, $\varphi(t) = k$ and $\delta = 0$, where *f*, *g* are injective as well, the result is clear.

Conclusions

The contraction $(\alpha,\beta,\varphi,\delta)$ applied for orbital-cyclic admissible and weakly compatible two pairs offer much more possibility for generalizations of existing results in extended cone metric spaces. Some of theorems cited in this paper can be interpreted as a particular case of our main result like Alqahtani et al. theorem and Haokip et al. theorem.

These results are true in *b*-cone metric spaces and in cone metric spaces since extended *b*-cone metrics are more general than such category of metric spaces.

As further study we suggest some applications of the main result in some special metric spaces generalized by cone metric spaces and in solving integral equations.

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