THE STUDY OF THE DYNAMICS OF THE QUAZI ZERO STIFFNESS (QZS) VIBRATION ISOLATOR WITH NUMERICAL INTEGRATION METHODS

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Abstract

Numerical methods of integration of differential equations give us a clearer and simpler picture of the dynamics and progress of the system, especially in cases where the system is unbalanced and jumps from one type of stability to another with the change of one of the parameters. The QZS vibration isolator is a simple case where numerical methods for solving ODE equations are quite favourable and efficient. In this study, a simplified Vibration Isolator model has been considered and the dynamics equations have been solved by means of numerical integration for different values of frequency and other parameters that affect the performance of the system assuming an inappropriate mass placed in system. Phase spaces, time series, and Poincare maps for the entire parameter space are presented. The chaoticity of the system has been studied by means of Lyapunov exponents and it has been concluded that the system is not chaos by considering different values of the control parameter. The presence of þeriodic windows"in the bifurcation diagram, which was simulated with a simple but special method, through Poincare maps, was also ascertained. Examining the system dynamics for a wide range of control parameters gives us a clear picture of the system performance.

Key words: *Numerical method, differential equation, phase space, time series, Lyapunov exponents.*

Përmbledhje

Metodat numerike të integrimit të ekuacioneve diferenciale na japin një tabllo më të qartë dhe të thjeshtë të dinamikës dhe të ecurisë së sistemit sidomos në rastet kur sistemi është i pa-ekuilibruar dhe kërcen nga një lloji stabiliteti në një tjetër me ndryshimin e njërit prej parametrave. Izolatori i vibrimeve QZS është një rast i thjeshtë ku metodat numerike për zgjidhjen e ekuacioneve ODE janë mjaft të favorshme dhe efikase. Në këtë studim është marrë në konsiderate një model i thjeshtuar Izolatori Vibrimesh dhe janë zgjidhur ekuacionet e dinamikës me anë të integrimit numerik për vlera të ndryshme të frekuencës dhe parametrave të tjerë të cilët ndikojnë në ecurinë e sistemit duke supozuar një masë të papërshtatshme të vendosur në sistem. Janë paraqitur hapësirat fazore, seritë kohore dhe hartat e Poincare për të gjithë hapësirën e parametrave. Është studiuar kaoticiteti i sistemit me anë të eksponentëve të Lyaponovit dhe është arritur përfundimi se sistemi nuk është kaos duke marrë në konsideratë vlera të ndryshme të parametrit të kontrollit. Gjithashtu është konstatuar prania e "dritareve periodike" në diagramën e bifurkacionit e cila është simuluar me një metodë të thjeshtë por të veçantë, nëpërmjet hartave të Poincare. Shqyrtimi i dinamikës së sistemit për një rang të gjerë të parametrave të kontrollit na jap një informacion të qartë të ecurisë së sistemit.

Fjalë kyçe: *Metoda numerike, ekuacione diferenciale, hapësira fazore, seritë kohore, eksponentët e Lyapunovit.*

Introduction

This study considers a simplified mechanical model of a vibration isolator and attempts to study the time course and stability of the system. The examined model consists of a differential equation (like the Duffing oscillator) where its analytical solution presents significant difficulties while the numerical integration greatly facilitates the way to the goal of the study. Using the 4th order Runge Kutta method turns out to be quite efficient and gives us enough information on what is required. However, the use of the ODE45 and ODE23 packages, whose basis is the same as the model cited above, further facilitates the work of solving the equations. After fitting the equations, the necessary simulations were performed for different values of the control parameters. The oscillation shown is indeed chaotic, so it is necessary to study whether the system reaches chaos. For this reason, by means of Lyapunov exponents, it has been seen that for different, but close, initial points, the trajectories do not diverge. To further see the performance of the system in relation to the control parameter (in this case, the frequency of the external exciter) we have simulated the bifurcation diagram which is extracted from a 3-dimensional system of Poincare maps.

1) Mathematical Modelling of a QZS Vibration Isolator

The vibration isolator has found wide use (Zhaozhao Ma et.al, 2022), (Snowdon, 1978) in the industrial and life fields undergoing a marked development in the theoretical point of view and more in its engineering context. However, in-depth analysis of the dynamics of the system and its behaviour for different scenarios of the impact of external factors on the system is necessary. Similar systems have been proposed by different researchers (Ivana Kovacica et.al, 2008), (Ngoc Vo, Tanh Le, 2022), (Meng, 2023) and have been analysed in detail for its behaviour in cases of external influences. The system we consider is the same as the model reviewed by Carrella et.al (2012).

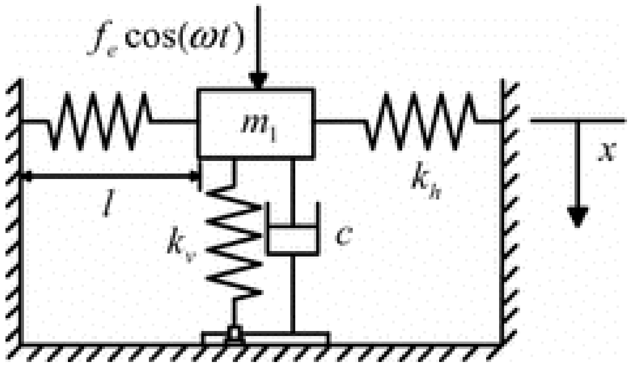


Figure 1. Vibration isolator, simplified model

In our case, the passive negative model of the vibration isolator QZS is considered as shown in figure 1. The figure shows the springs with the corresponding coefficients k_v and k_h while the coefficient of the piston is marked with C and we must take into account that F_d=cdx/dt which it means that as long as we are in equilibrium this force is not taken into consideration.

Writing the equations for the system under consideration:

$$\hat{F} = \hat{x} + 2\beta\left(1 - \frac{1}{\sqrt{\hat{x}^2(1-\hat{l}^2)+\hat{l}^2}}\right)\hat{x} \tag{1.1}$$

$$\hat{k} = (1+2\beta) - \frac{2\beta\hat{l}^2}{\left(\hat{x}^2\left(1-\hat{l}^2\right)+\hat{l}^2\right)^{\frac{3}{2}}} \tag{1.2}$$

Where $\hat{x} = \frac{x}{x_c}$, $\hat{F} = \frac{F}{k_v x_c}$, $\hat{l} = \frac{l}{l_0}$, $\beta = \frac{k_h}{k_v}$, $\hat{k} = \frac{k}{k_v}$, $x_c = (l_0^2 + l^2)^{1/2}$ at the static equilibrium position.

If the displacement amplitude from the equilibrium position is small, we can do the Taylor decomposition of equation 1.1 and arrive at the conclusion:

$$\hat{F}(\hat{x}) = \alpha\hat{x} + \gamma\hat{x}^3 \tag{1.3}$$

Where α and γ are given:

$$\alpha = 1 - 2\beta\frac{1-\hat{l}}{\hat{l}} \tag{1.4}$$

$$\gamma = \beta\frac{1-\hat{l}^2}{\hat{l}^3}$$

We have not referred to mathematical operations in the study because it is outside the scope of our paper. The mathematical model considering the case of placing a measure outside its predicted value (Yong Wang et.al, 2017) is given in the form:

$$m\ddot{x} + c\dot{x} + k_v\alpha(x+x_n) + \frac{\gamma k_v}{x_s^2}(x+x_n)^3 = mg + Fcos(\omega t) \tag{1.5}$$

Referring to studies (A.Carrella et.al, 2017; I. Kovacica et.al, 2008) the mathematical model of this system is given:

$$\ddot{x} + \zeta\dot{x} + b_1x + b_2x^3 = f_0 + f_1cos(\Omega\tau) \qquad 1.6$$

This model can be defined as the Duffing oscillator model (Jia, 2020), (Gatti, Ivana Kovacic , 2020), (Qu, 2020). Next, the values of the parameters were taken into consideration: δ = 0.15, $\hat{l}$= 0.667, β = 1, ζ = 0.025 (A.Carrella et.al, 2017, Yong Wang et.al, 2017).

2) Computer simulation, phase space and time series

The proposed mathematical model consists of differential equations of the second order, the solution of which will give us accurate information on the performance of the system. This mathematical model, like the Duffing model, presents difficulties in analytical integration, which is beyond the scope of this paper. The use of numerical integration methods with suitable approximations provides us with accurate and fast information about the performance of the system over time. The Euler method of numerical integration is the basic method from which other more advanced numerical integration methods such as the 4th order Runge Kutta method originate (Berwick, 2012). In this paper, the ODE 45 and ODE 23 algorithms provided by MATLAB (Berwick, 2012) and Octave are used, which is based on the 4th order Runge Kutta method (Said Ouala et.al, 2021). Using ODE45 greatly simplifies the numerical integration and follows a rigorous path of integration steps (Böttcher, 2021).

Next, iterations were performed up to 1000-time units with a step of 0.01 and with initial values (0,0,0) for different values of the control parameter Ω for the defined function of the case in question. For the equation to be suitable for the ODE45 algorithm executed in MATLAB, the necessary substitutions have been made to convert the second order differential equation into 3 first order differential equations as follows:

$$\dot{x} = y; \qquad 2.1$$

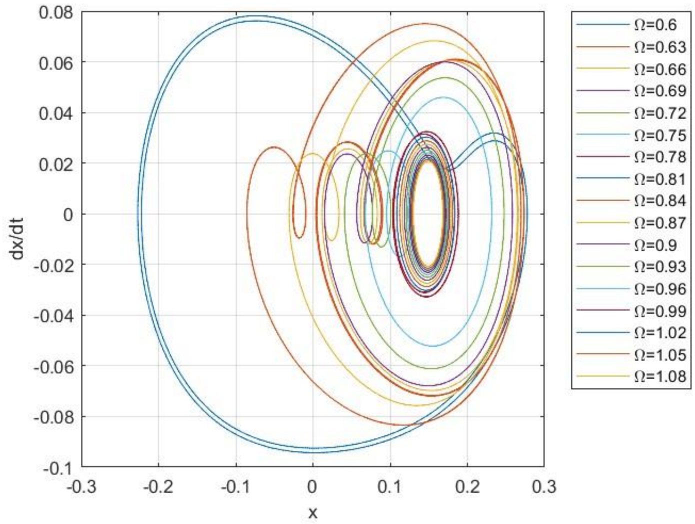
$$\dot{y} = -\zeta y - b_1x - b_2x^3 + f_0 + f_1cos(\Omega z) \qquad 2.2$$

$$\dot{z} = 1 \qquad 2.3$$

Where z represents time as a variable of the system of equations. In the following, the parameterized unit of time will be considered above the value of 200 because the system reaches stability in the limit cycle after a certain period of irregular oscillations, in cases where there is a limit cycle, while in cases where the behaviour is chaotic, the time indexing is done to clearly concluded in potential attractors. The same integrations are considered for different values of the control parameter to arrive at the study of the stability and performance of the system for a wider range of values of this parameter.

Figure 2 shows the phase spaces of the system oscillation for the control parameter from 0.6 to 1.08 as well as the progress of the phase space per time unit from 800 to 1000. The system does not undergo changes in time after joining the limit cycle and also we do not expect the system to arrive at chaos because there cannot be chaos in 2 size (Strogatz, 2003) but in the study we used the term "chaotic behaviour" not to define that the system is exactly chaotic but to characterize the quasi-chaotic and disordered behaviour.

Also, in the study we will prove that the model under consideration is not chaos. In Figure 2 we have presented the phase spaces for some cases of control parameters Ω which is frequence of oscillation. The first part (figure 2, a) represents a disordered system and as we will see below, the system undergoes chaotic behaviour and coincides with the case when f_1=0.02. The second part represents a more regular behaviour and as can be seen the system, through a bifurcation near the value 0.75, passes into a simple oscillator, which can also be observed in figure 2 and others, case which coincides with f_1=0.01.



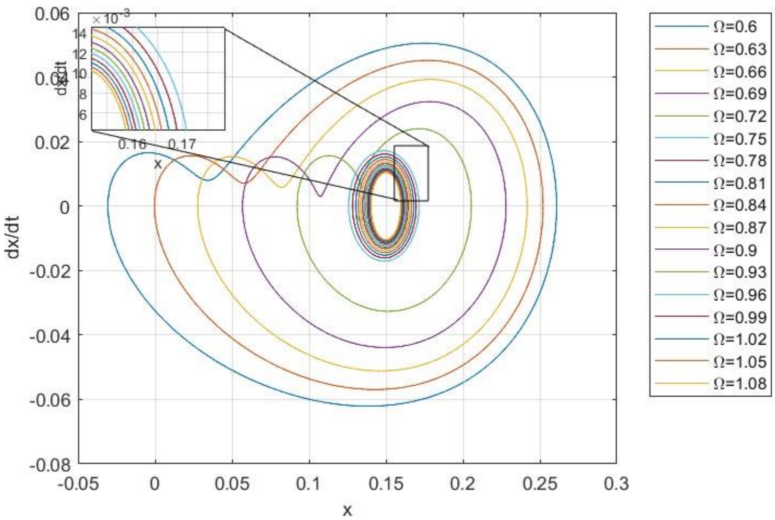


Figure 2. Phase spaces for values of Ω ranging from 0.6 to 1.08 for two different cases a) f_1=0.02, b) f_1=0.01.

In the following figures (Figure 3, 4, 5) we have presented the phase space and time series for some values of the Ω parameter, where we can more clearly analyse the oscillatory behaviour of the system under consideration. We see that the system goes through bifurcation processes from stable limit to chaotic behaviour and from chaotic behaviour to periodic oscillations, to two-periodic oscillations and to simple oscillators.

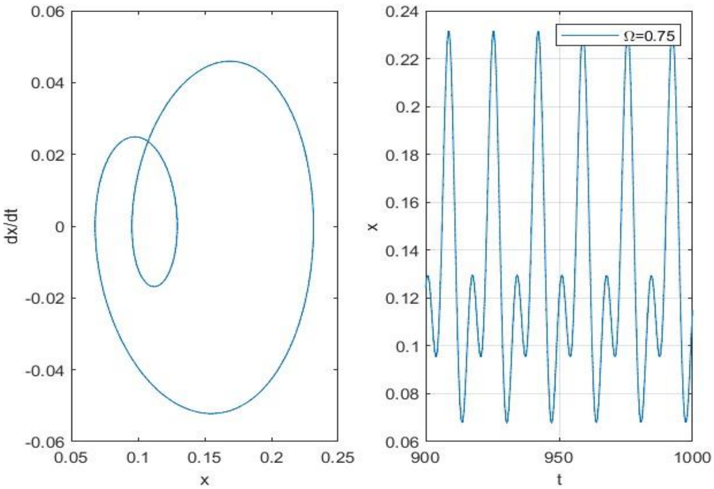


Figure 3. Phase spaces and time series for Ω=0.75

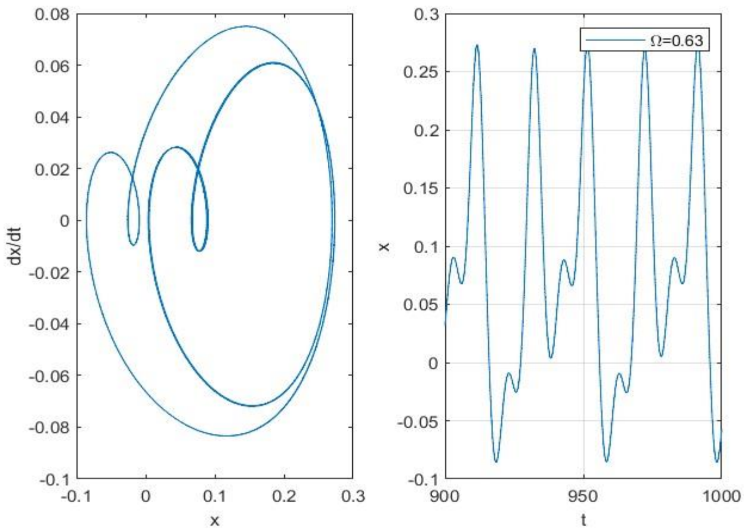


Figure 4. Phase spaces and time series for Ω=0.63.

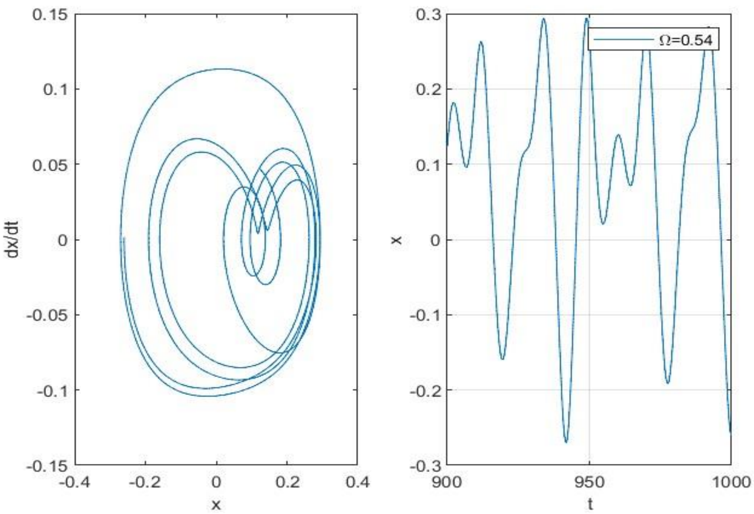


Figure 5. Phase spaces and time series for Ω=0.54

It is noted that near the value Ω=0.7 the system undergoes a clear bifurcation from multiperiodic oscillation to two-periodic oscillation, which we will see clearly in the bifurcation diagrams in the following figures.

3) Defining chaos

As we mentioned above, this system cannot be categorized as chaos because the minimum assessment does not meet the initial conditions of chaos, such as sensitivity to initial conditions (Ivana Kovacica et.al, 2008), (D. H. Rothman, 2022). The system for different initial conditions must have a positive Lyapunov exponent (at least one of them being positive) to be defined as chaos. If we look in figure 6 for none of the control parameter values, the divergence between the 2 trajectories for different initial values shows that the Lyapunov exponent is negative.

Considered both the dimensions presented x and dx/dt (assumed as an additional dimension) and the conclusion is the same. Initial values (0,0) and (0.01,0) were taken. In figure 5 we presented a more interesting model (Carrella et.al 2012), (Biswa, 2007) because for each value of t we have $dx=x_1(t)-x_2(t)$ and $dy=y_1(t)-y_2(t)$ were determined and the geometric sum was taken into consideration of divergence between trajectories. We also see that for some values of the frequency, the saturation of the divergence between the trajectories is reached faster than for some other values of the frequency.

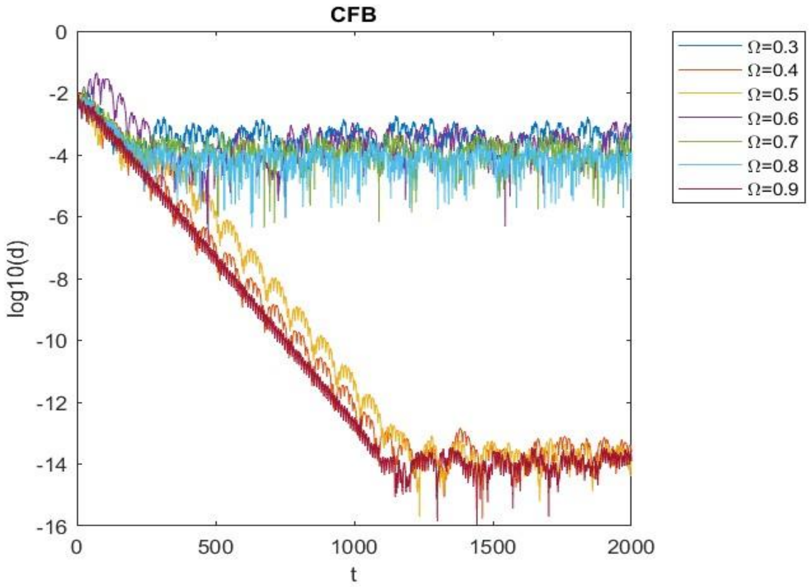


Figure 6. Determination of chaoticity by means of Lyapunov exponents for Ω=0.3 to 0.9.

4) Stability of solutions and bifurcations depending on parameters

The following figures show bifurcation diagrams and Poincare maps to explore the parameter space for different frequency values. All the values in figures 11 are obtained by choosing the values of x and y (where $y = \dot{x}$ according to the above equations) for a fixed value of time 2π/Ω. This fixed time value is modelled as a plane where the solutions of the equation are expected, and the points in figure represent the Poincare map (Barenghi, 2012).

Figure 9 present the Poincare maps for two values of the frequency Ω which represents the basic parameter for determining the Poincare plane or section. As we will explain below, we have extracted bifurcation diagrams from Poincare maps by performing numerical integrations and selecting numerical values for each Poincare map and plotting them for each frequency value.

In figure 7 we see the bifurcation diagram for a frequency spectrum from 0.3 to 1.1 where the solutions for each frequency value are displayed which gives us information about bifurcation oscillations and chaotic behaviour. In figure 7 and 8 (more detailed) we see that from the frequency value 0.3 to 0.5 the system has simple oscillations except for the case 0.41 to 0.45 where the system undergoes minor changes. We see that from a value of 0.5 to close to 0.55 the system offers chaotic behaviour within which there are periodic windows where the system departs from disordered and strange behaviour and approaches limit cycles. We see that above the value of 0.6 the system undergoes great changes in its oscillations, that it goes from 4 - periodic to 2 - periodic and the amplitude of the oscillation changes a lot.

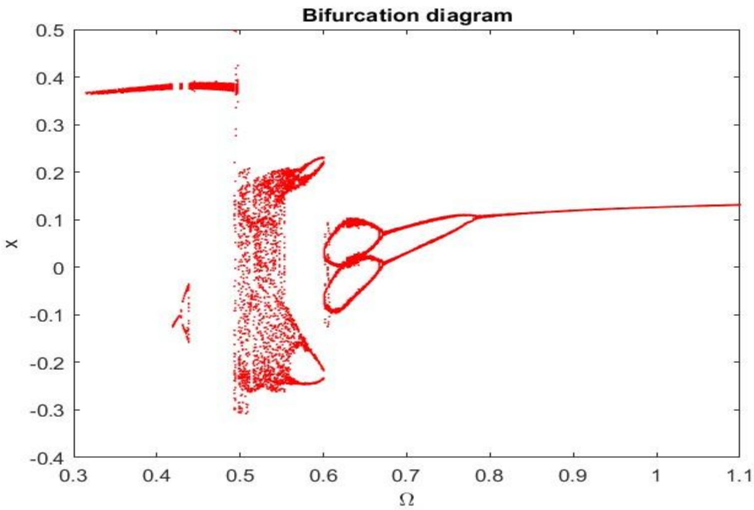


Figure 7. Bifurcation diagram for Ω ranging from 0.3 to 1.1 for the case where f_1=0.02

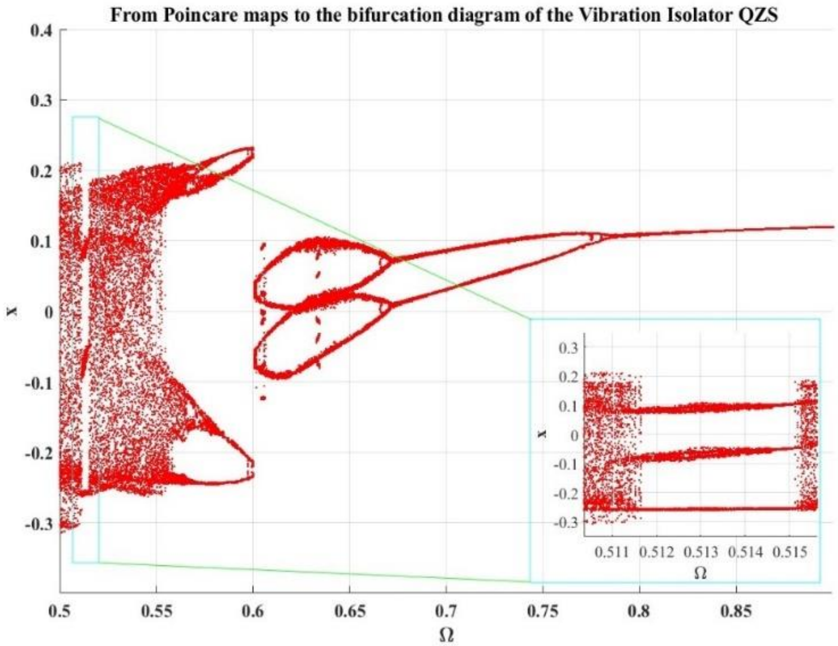


Figure 8. Bifurcation diagram for Ω ranging from 0.5 to 0.9 for the case when f_1=0.02 and finding periodic windows.

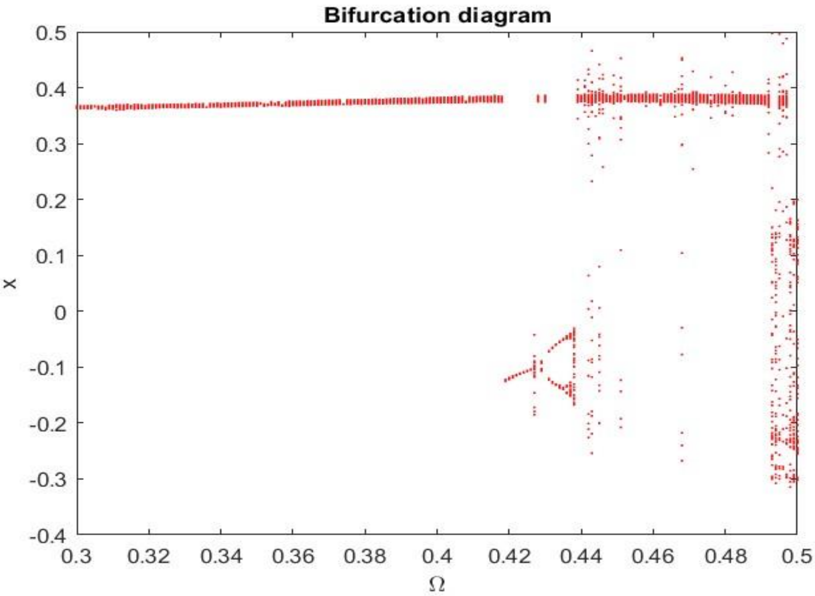


Figure 9. Bifurcation diagram for Ω ranging from 0.3 to 0.5 for the case where $f_1=0.02$.

In figure 10 we see the bifurcation diagrams for $f_1 = 0.01$ and 0.015, respectively, which show less disorder and chaotic behaviour than the case when $f_1=0.02$. However, we see that the case when $f_1=0.015$ lies in an intermediate case between the diagrams for $f_1 = 0.01$ and $f_1 = 0.02$. In all cases Hopf bifurcations and stable oscillations according to a multi-periodic limit cycle 1 appear.

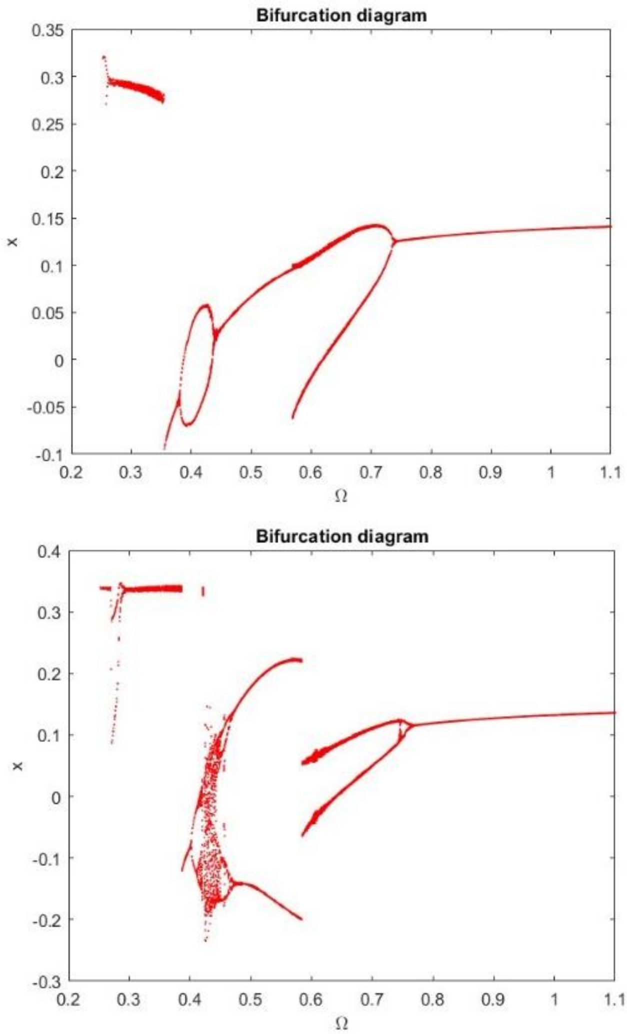


Figure 10. Bifurcation diagram where omega ranges from 0.2 to 1.1 for f_1=0.01 and 0.015.

Figure 11 shows the Poincare maps for Ω=0.5 and for 0.55, whereas seen the system undergoes extreme changes, but since the system, referring to the bifurcation diagrams, is quite sensitive to the frequency of the exciting oscillation, it is expected that the Poincare maps have great divergences between them.

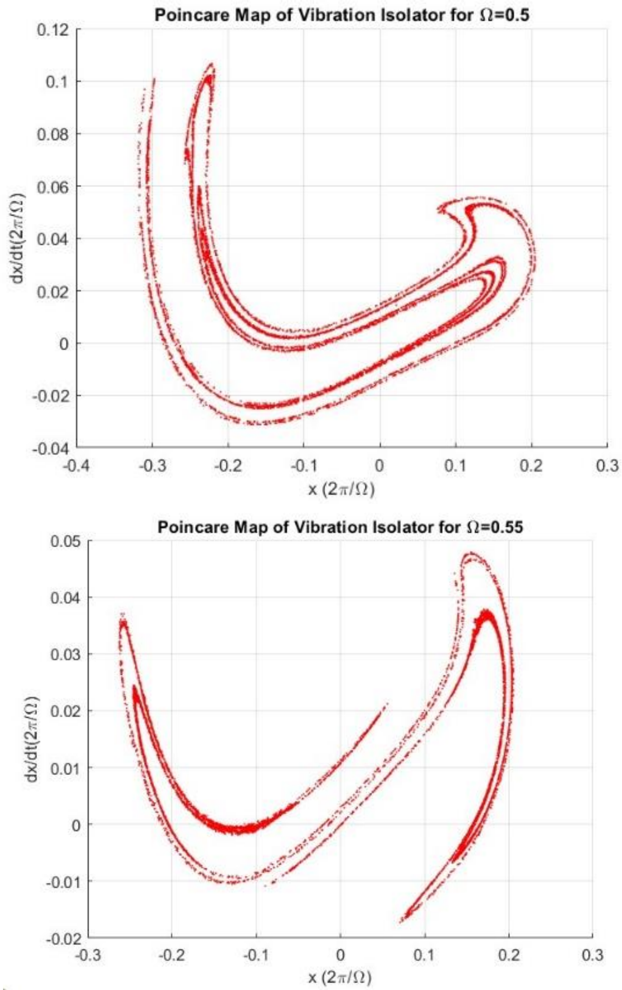


Figure 11. Poincare map for Ω=0.5 and 0.55

5) From Poincare maps to the bifurcation diagrams

As mentioned, it is convenient and easier to convert Poincare maps into bifurcation diagrams by manipulating MATLAB commands. In the following figures, the graphical illustration is given by means of 3D figures with parallel planes for each value of the control parameter.

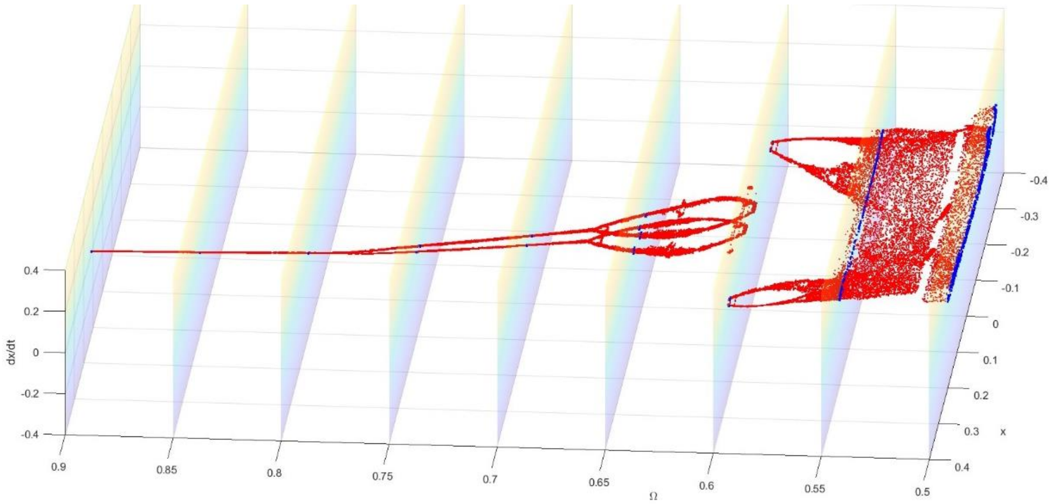


Figure 12. 3D illustration of the bifurcation diagram simulation method from Poincare maps.

To get the bifurcation diagram from the Poincare map we have made a 90-degree rotation of the 3-dimensional figure, where the Poincare maps from the 2-dimensional map will appear simply as dots on a line. The simulation of the Poincare maps for closer values of the control parameter gives us a clearer view of the bifurcation diagram.

Conclusions

The presented study provides information on the performance of the vibration isolator system for a wide range of excitation frequency values. The case when f_1 is 0.01 the system is more stabilized while for the value 0.02 the system exhibits chaotic behaviour. The degree of chaoticity is studied by means of Lyapunov exponents, according to which, since they are negative for different values of the frequency, the system cannot be accepted as chaos, but behaves quite close to it.

The system undergoes bifurcations for parameterized values of frequency 0.6, 0.68 and 0.78 approximately. Poincare maps for the 0.5 and 0.55 frequency values are also presented, and further we presented the method of simulating bifurcation diagrams from Poincare maps. All simulations were performed with MATLAB, where the solution of the differential equations was performed with ODE45 and ODE23 which gave high efficiency in the numerical integration of the system.

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