WELL-POSEDNESS AND VARIATIONAL ANALYSIS OF AN ELLIPTIC PARTIAL DIFFERENTIAL PROBLEM

REDJOLA MANAJ[1], BESIANA ÇOBANI[2]

[1]Department of Computer Engineering, Faculty of Architecture and Engineering, Epoka University, Tirana, Albania

[2]Department of Mathematics, Faculty of Natural Sciences,

University of Tirana, Albania

e-mail: rmanaj@epoka.edu.al

Abstract

This work examines a model elliptic boundary value problem that models anisotropic diffusion inside the unit disc. Such problems arise in a wide range of physical application. The differential operator that we consider in this work includes a mixed derivative term and a coefficient which depends on the spatial variable, which makes this operator particularly interesting in the theory of elliptic partial differential equations. We begin with the weak formulation of the problem, which is studied in the Sobolev space $H_0^1(\Omega)$, and we analyze the associated bilinear form. By using several fundamental results from Sobolev spaces, including the Poincaré inequality, continuity of linear functionals, and the properties of uniformly elliptic matrices, we have shown that our bilinear form satisfies the condition to be both continuous and coercive. These analytical results enable the direct application of the Lax–Milgram theorem, ensuring the existence and uniqueness of a weak solution for any square-integrable source term. A complete derivation of the corresponding strong form of the partial differential equation is provided by carefully performing all integration-by-parts steps, explicitly tracking contributions from variable coefficients and mixed derivatives. The methodology presented here forms a foundation for further analytical investigations and supplies the theoretical framework necessary for numerical approximation techniques.

***Key words**: variational formulation, Lax-Milgram Theorem, bilinear form, well-posedness.*

Përmbledhje

Ky punim shqyrton një problem të ekuacioneve eliptike me kushte kufitare i cili përshkruan difuzionin anizotropik brenda rrethit njësi. Probleme të tilla kanë një gamë të gjerë aplikimesh në fizikë. Operatori diferencial i konsideruar në këtë punë përfshin një term me derivat të përzier dhe një koeficient që varet nga variabla hapësinore y*, çka e bën këtë një rast interesant në fushën e ekuacioneve diferenciale eliptike. Ne fillojmë me formulimin e dobët të problemit, i cili studiohet në hapësirën Sobolev, dhe analizojmë formën bilineare të lidhur me të. Duke përdorur njohuri të rëndësishme nga teoria e hapësirave Sobolev, përfshirë inekuacionin Poincaré-së, vazhdimësinë e funksionaleve lineare dhe vetitë e matricave uniformisht eliptike, tregojmë se forma jonë bilineare plotëson kushtet për të qenë njëkohësisht e vazhdueshme dhe coercive. Këto rezultate analitike mundësojnë zbatimin e drejtpërdrejtë të Teoremës së Lax–Milgram, e cila garanton ekzistencën dhe unicitetin e një zgjidhjeje të dobët për çdo term nga* $L_2(\Omega)$*. Më tej, ne nxjerrim plotësisht formën e plotë të ekuacionit diferencial me derivate të pjesshme, duke kryer me kujdes të gjitha hapat e integrimit me pjesë, duke ndjekur kontributet nga koeficientët variabël dhe derivatet e përziera. Metodologjia e paraqitur këtu shërben si bazë për kërkime të mëtejshme analitike dhe ofron kornizën teorike të nevojshme për teknikat numerike.*

***Fjalë kyçe:** formulim variacional, Teorema e Lax–Milgram, formë bilineare, mirë–pozueshmëri.*

Introduction

Elliptic boundary value problems with variable coefficients serve as fundamental models in physics and engineering, describing phenomena such as heat conduction in non-homogeneous material, or fluid flow through porous media. A powerful method in studying such problems is the variational analysis, where solutions are seen as functions satisfying a weak formulation, which is derived from a suitable bilinear form on a Hilbert space.

In this work we are going to examine a non-standard bilinear form, which is defined one the Sobolev Space $H_0^1(\Omega)$, over the unit disc $\Omega = \{(x,y); x^2+y^2<1\}$:

$$\langle u,v\rangle_\diamond = \int_\Omega \left[\, u_x v_x + 2\, u_y v_y + \mathrm{y}\left(u_x v_y + u_y v_x\right)\right] dxdy$$

Our analysis is focused in two main parts. Firsly, we establish that $\langle .,.\rangle_\diamond$ defines an inner product equivalent to the standard H_0^1- norm, thereby equipping $H_0^1(\Omega)$ with a new Hilbertian structure which is more adapt to the nature of the problem.

The second part is working with the elliptic form by showing the well-posedness of the problem: for a given function $f \in L^2(\Omega)$, we apply the Lax-Milgram theorem to establish the existence and uniqueness of a weak solution $u \in H_0^1(\Omega)$, satisfying the variational equation

$$\langle u,v\rangle_\diamond = \int_\Omega fvdx, \quad \text{for all } v \in H_0^1(\Omega),$$

and we derive the corresponding second-order elliptic boundary value problem explicitly.

The novelty of this work lies in the explicit variational treatment of the elliptic operator with mixed derivatives on the unit disc, including a detailed coercivity analysis via eigenvalue estimates.

Methodology

Throughout this work we recall several theoretical tools that are essential for the analysis and will be used to establish our results. These tools are summarized in this section.

Definition 1. (Robert A. Adams, 2003)

For an open set $\Omega \in \mathbb{R}^n$,

$$H^1(\Omega) = \{u \in L^2(\Omega) : \partial_{x_i} x \in L^2(\Omega) for\ all\ i\}$$

It is equipped with the norm:

$$\|u\|^2_{H^1(\Omega)} = \|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}$$

The space $H_0^1(\Omega)$ is the closure of $C_c^\infty(\Omega)$ under the H^1 −norm.

Definition 2 (Salsa, 2008)

A linear functional L on a Banach space $(H, \|.\|_H)$ is continuous if and only if it is bounded, i.e., if there $\exists C < \infty$, such that

$$|L(v)| \leq C\|v\|_H \text{ }, \forall v \in H$$

Definition 3 (Poincaré-Friedrichs inequality) (Sayas, 2019)

If $\Omega \subset \mathbb{R}^{d-1} \times (a, b)$ (or, more generally, if Ω is bounded in at least one direction, then:

$$\|u\|_\Omega \leq \frac{b-a}{2}\|\nabla u\|_\Omega, \forall u \in H_0^1(\Omega)$$

Definition 4 (David Gilbarg, 1998)

A matrix field $A(x)$ is uniformly elliptic if there exists $\lambda > 0$, such that:

$$A(x)\xi \cdot \xi \geq \lambda|\xi|^2, \forall \xi \in \mathbb{R}^n, \forall x \in \Omega$$

Definition 5 (Bressan, 2012)

Let H be a Hilbert space over the reals and let $B: H \times H \rightarrow \mathbb{R}$ be a continuous bilinear functional. This means that

$$B[au + bu', v] = aB[u, v] + bB[u', v]$$

$$B[u, av + bv'] = aB[u, v] + bB[u, v']$$

$$|B[u, v]| \leq C\|u\|\|v\|$$

For some constant C, and all $u, u', v, v' \in H, a, b \in \mathbb{R}$.

Definition 6 (Chipot, 2009)

Let H be a Hilbert space over the reals and let $B: H \times H \rightarrow \mathbb{R}$ be a continuous bilinear functional B is strictly positive definie (coercive), i.e., there exists a constant $\beta > 0$ such that

$$B[u, u] \geq \beta\|u\|^2 \; for \; all \; u \in H$$

Theorem 1 (Lax-Milgram) (Bressan, 2012), Let H be a Hilbert space over the reals and let $B: H \times H \to \mathbb{R}$ be a continuous bilinear functional. Furthermore the bilinear form is coercive.

Then, for every $f \in H$, there exists a unique $u \in H$ such that

$$B[u,v] = (f,v)\ for\ all\ v \in H.$$

Moreover,

$$\|u\| \leq \beta^{-1}\|f\|.$$

Analysis and discussions

The problem is set in the domain $\Omega = \{(x,y); x^2 + y^2 < 1\}$ which denotes the unit disc in $\mathbb{R}^2$. We work in the Sobolev space $H_0^1(\Omega)$, which is defined as the closure of $C_c^\infty(\Omega)$ with respect to the norm:

$$\|u\|_{H^1(\Omega)} = \left(\int_\Omega (|\nabla \mathrm{u}|^2 + u^2)dxdy\right)^{1/2}$$

By the Poincaré inequality, for all $u \in H_0^1(\Omega)$,

$$\|u\|_{L^2(\Omega)} \coloneqq \|\nabla u\|_{L^2(\Omega)}$$

Is equivalent to $\|u\|_{H^1(\Omega)}$on $H_0^1(\Omega)$, so we equip it with this norm.

The bilinear form is defined as:

$$B[u,v] = \langle u,v\rangle_\diamond = \int_\Omega [\,u_x v_x + 2\,u_y v_y + \mathrm{y}(u_x v_y + u_y v_x)]dxdy$$

Which can be written equivalently in the form:

$$B[u,v] = \int_\Omega A(y)\nabla u \cdot \nabla v\ dxdy\,, where\ A(y) = \begin{pmatrix} 1 & y \\ y & 2 \end{pmatrix}$$

Firstly, we must prove that $\langle .,.\rangle_\diamond$ is indeed an inner product on $X = H_0^1(\Omega)$, which makes X a Hilbert space.

An inner product must satisfy the following properties (Evans, 2010):

1. Linearity: $\langle au + bv, w\rangle_\diamond = a\langle u,w\rangle_\diamond + b\langle v,w\rangle_\diamond$ for all $u,v,w \in X$ and all scalars a,b.

This comes from the fact that both the integral and partial derivatives are linear operators. So, we have the following result:

$$\langle au+bv,w\rangle_\circ=\int_\Omega\big[(au+bv)_x w_x+2\,(au+bv)_y w_y$$
$$+\,\mathrm{y}\big((au+bv)_x w_y+(au+bv)_y w_x\big)\big]dxdy$$
$$=\int_\Omega\Big[(au_x+bv_x)w_x+2\,\big(au_y+bv_y\big)w_y$$
$$+\,\mathrm{y}\Big((au_x+bv_x)w_y+\big(au_y+bv_y\big)w_x\Big)\Big]\,dxdy$$
$$=a\int_\Omega\big[\big(u_x w_x+2u_y w_y\big)+\mathrm{y}\big(u_x w_y+u_y w_x\big)\big]dxdy+$$
$$+\,b\int_\Omega\big[\big(v_x w_x+2\,v_y w_y\big)+\mathrm{y}\big(v_x w_y+v_y w_x\big)\big]dxdy\;=$$
$$=a\langle u,w\rangle_\circ+b\langle v,w\rangle_\circ.$$

2. Symmetry: $\langle u,v\rangle_\circ=\langle v,u\rangle_\circ$ for all u, $v\in X$.

$$\langle u,v\rangle_\circ=\int_\Omega\big[\,u_x v_x+2\,u_y v_y+\mathrm{y}\big(u_x v_y+u_y v_x\big)\big]dxdy=$$
$$=\int_\Omega\big[v_x\,u_x+2\,v_y u_y+\mathrm{y}\big(v_x u_y+v_y u_x\big)\big]dxdy=\langle v,u\rangle_\circ$$

3. Positivity: $\langle u,u\rangle_\circ\geq 0$ for all $u\in X$, with equality if and only if $u=0$

$$\langle u,u\rangle_\circ=\int_\Omega\big[\,u_x^2+2\,u_y^2+y\big(u_x u_y+u_y u_x\big)\big]dxdy$$
$$=\int_\Omega\big[\,u_x^2+2\,u_y^2+2\mathrm{y}u_x u_y\big]dxdy$$
$$=\int_\Omega\Big[\,\big(u_x+yu_y\big)^2+u_y^2(2-y^2)\Big]\,dxdy\geq 0$$

This result holds since domain is the unit disc.
Furthermore, let's suppose that $\langle u,u\rangle_\circ=0$, so the last integral is 0 only if both terms are 0, $u_x+yu_y=0, and\ u_y=0\ if$ and only if $u_x=0, u_y=0\rightarrow u=0$.
Now we use the inner product to define the norm $\|u\|_\circ=\sqrt{\langle u,u\rangle_\circ}$
From the identity $u_x^2+2\,u_y^2+2\mathrm{y}u_x u_y=\big(u_x+yu_y\big)^2+u_y^2(2-y^2)$ we have the bounds: $u_x^2+2\,u_y^2+2\mathrm{y}u_x u_y\leq 3(u_x^2+u_y^2)$ since $|y|\leq 1$
Integrating over Ω yields

$$\|\nabla u\|^2_{L^2(\Omega)} \leq \|u\|^2_\diamond \leq 3\|\nabla u\|^2_{L^2(\Omega)}$$

As a result, we have that $\|.\|_\diamond$ is equivalent to the standard norm. Since $H_0^1(\Omega)$ is complete under $\|\nabla u\|_{L^2(\Omega)}$ it is also complete under this norm as well. Hence $H_0^1(\Omega)$ is a Hilbert space.

With the Hilbert space structure of $(H_0^1(\Omega), \langle .,. \rangle_\diamond)$ already shown, we now continue with the associated variational problem. Given $f \in L^2(\Omega)$ we seek for a unique $u \in H_0^1(\Omega)$ such that

$$\langle u, v\rangle_\diamond = \int_\Omega f v dx dy \quad \forall u \in H_0^1(\Omega)$$

For us to show the well-posedness of this weak formulation we have to show that it satisfies the conditions of the Lax-Milgram Theorem.

From our problem we can take as the required bilinear form

$$B[u, v] = \langle u, v\rangle_\diamond \text{ and } F(v) = \int_\Omega f v dx\text{dy}.$$

From Definition 5 we have to show that it satisfies the condition to be continuous, we have to show the existence of a constant C, such that

$$|B[u, v]| \leq C\|u\|\|v\|$$

By using the Cauchy-Schwarz ineqaulity and the bound $|\text{y}| \leq 1$ in Ω we have:

$$\begin{aligned}|B[u, v]| = |\langle u, v\rangle_\diamond| &= \left|\iint_\Omega [\, u_x v_x + 2\, u_y v_y + \text{y}(u_x v_y + u_y v_x)] dx dy\right| \\ &\leq \int_\Omega |u_x v_x| + 2\,|u_y v_y| + |\text{y}||(u_x v_y + u_y v_x)| dx dy \\ &\leq 3\int_\Omega \left(|u_x|^2 + |u_y|^2\right)^{1/2} \left(|v_x|^2 + |v_y|^2\right)^{1/2} dx dy \\ &\leq 3\,\|\nabla u\|_{L^2} \cdot \|\nabla v\|_{L^2}\end{aligned}$$

Using $\|\nabla u\|^2_{L^2(\Omega)} \leq \|u\|^2_\diamond \leq 3\|\nabla u\|^2_{L^2(\Omega)}$ (shown above), we have $\|\nabla u\|_{L^2} \leq \|u\|_\diamond$ and similarly we have for v. Hence

$$|B[u, v]| \leq 3\|u\|\|v\|$$

From definition 6 in order to show that this bilinear form is coercive we have to show that there exists a constant $\alpha > 0$ such that

$$B[u,u] \geq \alpha \|\nabla u\|^2_{L^2(\Omega)} \quad \forall\, u \in H^1_0(\Omega)$$

$$\begin{aligned} B[u,u] &= \int_\Omega \left[\, u_x^2 + 2\, u_y^2 + y\left(u_x u_y + u_y u_x\right)\right] dxdy \\ &= \int_\Omega \left[\, u_x^2 + 2\, u_y^2 + 2y u_x u_y\right] dxdy \\ &= \int_\Omega (\nabla u)^T A(y) \nabla\, u dxdy \end{aligned}$$

with $A(y) = \begin{pmatrix} 1 & y \\ y & 2 \end{pmatrix}$.

So, we have that at each point $(x,y) \in \Omega$ the integrand is:

$$Q_y(\xi) := \xi^T A(y) \xi = \xi_1^2 + 2\xi_2^2 + 2y\xi_1\xi_2$$

Where $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$. We want a lower bound of $Q_y(\xi)$ in terms of $|\xi|^2, \forall |y| \leq 1$.

That is, we want a constant $\alpha > 0$ such that

$$\xi^T A(y) \xi \geq \alpha |\xi|^2, \forall \xi \in \mathbb{R}^2, \qquad \forall |y| \leq 1$$

Analyzing matrix A, which is a symmetric matrix, we know that it is diagonalizable with real eigenvalues, and furthermore for any $\xi \neq 0$,

$$\frac{\xi^T A(y) \xi}{|\xi|^2}$$

Takes values between the smallest and the largest eigenvalues of $A(y)$.

So, after finding the eigenvalues of this matrix we have the following result:

$$\lambda_{max}(y) = \frac{3 + \sqrt{1+4y^2}}{2},\ \lambda_{min}(y) = \frac{3 - \sqrt{1+4y^2}}{2},$$

Now we need a lower bound on $\lambda_{min}(y), \forall |y| \leq 1$.

Since we are working in the unit disc we have the following result:

$y^2 \in [0,1]$, so, the term $1 + 4y^2 \in [1,5] \rightarrow \sqrt{1+4y^2} \in \left[1, \sqrt{5}\right]$

The map

$$\phi(t) = \frac{3-t}{2}$$

is decreasing in t. So, the smallest value of $\lambda_{min}(y) = \phi\left(\sqrt{1+4y^2}\right)$ occurs when $\sqrt{1+4y^2}$ is largest, i.e. when $y^2 = 1$.

Hence,

$$\lambda_{min}(y) \geq \frac{3-\sqrt{5}}{2} := \alpha > 0, \forall |y| \leq 1$$

Furthermore, we have a uniform ellipticity constant

$$\alpha = \frac{3-\sqrt{5}}{2}$$

We will use this part for coercivity, since we have for each fixed y and any vector ξ:

$$\xi^T A(y)\xi \geq \lambda_{min}(y)|\xi|^2 \geq \alpha|\xi|^2$$

By applying this with $\xi = \nabla u(x,y)$ pointwise we get:

$$(\nabla u)^T A(y)\nabla u \geq \alpha|\nabla u|^2 \quad \forall (x,y) \in \Omega$$

Integrating over Ω we get:

$$B[u,u] = \int_\Omega (\nabla u)^T A(y)\nabla u \, dxdy \geq \alpha \int_\Omega |\nabla u|^2 dxdy = \alpha \left|\left|\nabla u\right|\right|_{L^2(\Omega)}^2$$

Which is exactly what we needed for coercivity.

The last condition that we must check is the continuity of the linear functional:

$$L(v) = \int_\Omega fv \, dxdy$$

Which means we have to show existence of a constant $C > 0$, such that $\forall v \in H_0^1(\Omega)$:

$$|L(v)| \leq C||v||_H$$

But since, by using the Poincaré inequality (Evans, 2010) we get:

$$|L(v)| \leq \|f\|_{L^2}\|v\|_{L^2} \leq C_P \, \|f\|_{L^2}\|\nabla v\|_{L^2} = C\|v\|_H$$

Now we have all the conditions of Lax-Milgram theorem satisfied, and as a result we have showin that there exist a unique $u \in H_0^1(\Omega)$ such that:

$$B[u,v] = L(v), \quad \forall v \in H_0^1(\Omega)$$

This is the weak solution to the PDE.

After showing all this, which guarantees the existence of the weak solution we have to find the corresponding elliptic PDE that this weak solution actually satisfies.

For this part, we choose test fucntion with compact support $v \in C_c^\infty(\Omega)$. Since these function v vanishes on $\partial\Omega$, all boundary terms in integration by parts disappear.

By integrating now each term by parts we compute the following result.

$$\int_\Omega u_x v_x \, dxdy = -\int_\Omega u_{xx} v dxdy$$

For this we have used the fact that:

$$\frac{\partial}{\partial x}(u_x v) = u_{xx} v + u_x v_x$$

And therefore integrating over Ω we have:

$$\int_\Omega \partial_x (u_x v) \, dxdy = \int_\Omega (u_{xx} v + u_x v_x) dxdy$$

By applying divergence theorem:

$$\int_\Omega \partial_x (u_x v) dxdy = \int_{\partial\Omega} u_x v \, n_x ds$$

But $v = 0 \; on \; \partial\Omega$, so the boundary term is zero.

Therefore we get:

$$\int_\Omega u_x v_x \, dxdy = -\int_\Omega u_{xx} v dxdy$$

Similarly we have:

$$\int_\Omega 2u_y v_y dxdy = -2\int_\Omega u_{yy} v dxdy$$

For the last two terms we have:

$$\int_\Omega y u_x v_y dxdy = -\int_\Omega (y u_{xy} + u_x) v dxdy$$

$$\int_\Omega y u_y v_x dxdy = -\int_\Omega y u_{xy} v dxdy$$

By puttinng everything together now we can find the equation in the following form:

$$B[u,v] = \int_{\Omega} [\, u_x^2 + 2\, u_y^2 + y(u_x u_y + u_y u_x)]dxdy$$

$$B[u,v] = -\int_{\Omega} u_{xx} v dxdy - 2\int_{\Omega} u_{yy} v dxdy - \int_{\Omega} (y u_{xy} + u_x) v dxdy - \int_{\Omega} y u_{xy} v dxdy$$

Thus we have:

$$B[u,v] = -\int_{\Omega} (u_{xx} + 2u_{yy} + 2y u_{xy} + u_x) v dxdy$$

From the weak formulation we have $B[u,v] = \int_{\Omega} fv$

And since this holde for all $v \in C_c^{\infty}\,(\Omega)$, we get the PDE:

$$-u_{xx} - 2u_{yy} - 2y u_{xy} - u_x = f \;\; in\; \Omega$$

The homogeneous Dirichlet boundary condition is implicitly imposed by the choice of the space $H_0^1(\Omega)$

$$u = 0 \;\; on\; \partial\Omega$$

Collecting these reaults, we obtrain the following elliptic boundary value problem:

$$\begin{cases} -u_{xx} - 2u_{yy} - 2y u_{xy} - u_x = f, & (x,y) \in \Omega \\ u = 0 & (x,y) \in \partial\Omega \end{cases}$$

Conclusions

In this work we have provided the complete variational and functional analytic study of an elliptic boundary value problem posed on the unit disc. By formulating the problem in Sobolev space $H_0^1(\Omega)$, we have analyzed in a rigorous way the bilinear form which is associated with the operator, and it satisfies all the requirements of continuity and coercivity. We have shown continuity by using classical inequalities like Poincaré and Cauchy-

Schwarz. As for coercivity we have used the eigenvalues of the matrix obtained by bilinear form, which remain uniformly positive in the domain.

We have applied Lax-Milgram theorem, and we guaranteed the well posedness of the problem, which ensures stability and robustness with respect to the data. Throughout integration by parts, we obtained the explicit PDE which is satisfied by u.

The result achieved provide both a rigorous theoretical understanding of the operator, from which numerical approximation methods (Monk, 1991) can be developed. Consequently, this work serves for future investigations and computation studies of even more complex form.

References

Bressan, A. (2012). Lecture Notes on Functional Analysis with Applications to Linear Partial Differential Equations. American Mathematical Society.

Chipot, M. (2009). Elliptic equations; an introductory course. Birkhäuser Advance Texts.

David Gilbarg, N. S. (1998). Elliptic Partial Differential Equation of Second Order. Springer.

Evans, L. C. (2010). Partial Differential Equations. Rhode Island: American Mathematical Society.

Monk, P. B. (1991). A Mixed Method for Approximating Maxwell's Equation. Society for Industrial and Applied Mathematics .

Robert A. Adams, J. J. (2003). Sobolev Spaces. Elsevier.

Salsa, S. (2008). Partial differential equations in action: from modelling to theoru. Springer.

Sayas, F.-J. (2019). Variational Techniques for Elliptic Partial Differential Equations. CRC Press.