MENGER PM SPACES OF HYPERBOLIC TYPE AND FIXED POINT RESULTS FOR FUNDAMENTALLY NONEXPANSIVE MAPPINGS

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Abstract

In this paper, we focus on Menger probabilistic metric spaces of hyperbolic type, extending the notion of hyperbolic metric spaces introduced by W. A. Kirk to the probabilistic framework initiated by Karl Menger and further developed by B. Schweizer and A. Sklar. We explore the fundamental properties of these spaces and provide illustrative examples, including a concrete construction on the positive real line equipped with the Lukasiewicz t-norm. This example shows how hyperbolic-type convexity can be formulated in the context of a Menger probabilistic metric space. A central result of this paper is the proof of a key lemma concerning the asymptotic behavior of two bounded sequences in Menger probabilistic metric spaces of hyperbolic type. As a main application, we introduce the concept of fundamentally nonexpansive mappings and prove a fixed point theorem for such mappings defined on nonempty compact convex subsets of Menger probabilistic metric spaces of hyperbolic type. The obtained results extend classical fixed point theorems from metric and hyperbolic spaces to the probabilistic setting and contribute to the development of nonlinear analysis in probabilistic metric spaces.

Key words**:** *Menger probabilistic metric space, convex structure, hyperbolic type, fixed point.*

Përmbledhje

Në këtë punim, ne përqendrohemi në hapësirat metrike probabilitare Menger të tipit hiperbolik, duke zgjeruar nocionin e hapësirave metrike të tipit hiperbolik të prezantuara nga W. A., Kirk në kornizën probabilitare të filluar nga Karl Menger dhe të zhvilluar më tej nga Schweizer dhe Sklar. Ne shqyrtojmë vetitë themelore të këtyre hapësirave dhe japim shembuj ilustrues, duke përfshirë një ndërtim konkret në drejtëzën reale pozitive të pajisur me t-

normën e Lukasiewicz-it. Ky shembull tregon se si konveksiteti i tipit hiperbolik mund të formulohet në kontekstin e një hapësire metrike probabilitare Menger. Një rezultat qendror i këtij punimi është vërtetimi i një leme kyçe në lidhje me sjelljen asimptotike të dy vargjeve të kufizuara në hapësirat metrike probabilitare Menger të tipit hiperbolik. Si zbatim kryesor, ne prezantojmë konceptin e pasqyrimeve thelbësisht jo-ekspansive dhe provojmë një teoremë të pikës fikse për pasqyrime të tilla të përcaktuara në një nënbashkësi konvekse kompakte dhe jo-boshe të hapësirave metrike probabilitare të Menger-it të tipit hiperbolik. Rezultatet e përftuara zgjerojnë teoremat klasike të pikave fikse nga hapësirat metrike dhe të tipit hiperbolik në kuadrin probabilistik dhe kontribuojnë në zhvillimin e analizës jolineare në hapësirat metrike probabilitare.

Fjalë kyçe: *hapësirë metrike probabilitare Menger, strukturë konvekse, tipi hiperbolik, pikë fikse.*

Introduction

Banach (1922) introduced the Banach contraction principle in metric spaces, which has since become a cornerstone in the development of fixed point theory across various branches of mathematical analysis, and many other fields. Over the past few decades, the concept of metric spaces has been extended to capture uncertainty and stochastic behavior in more flexible frameworks.

Menger (1942) introduced the theory of probabilistic metric spaces, marking a significant advancement. Probabilistic metric spaces offer a probabilistic generalization of traditional metric spaces, where the distance between pairs of elements is represented by distribution functions rather than non-negative real numbers. This probabilistic framework allows for a richer description of uncertainty in the geometry of spaces. This work was further elaborated by Schweizer & Sklar (1983), who explored its properties, including topology, sequence convergence, mapping continuity, and space completeness. The flexibility inherent in these spaces allows for the extension of the contracting mapping principle in various non-equivalent ways. A Menger space is a specific type of probabilistic metric space that employs a t-norm to define the triangular inequality. This concept is crucial in stochastic research.

The first fixed point result in probabilistic metric (PM) spaces was achieved by Sehgal and Bharucha–Reid (1972), who further generalized the Banach contraction principle within these spaces. This laid the foundation for subsequent researchers to extensively explore fixed point theorems in the context of probabilistic distances.

Furthermore, Hadžić (1987) introduced the notion of a convex structure for sets in Menger probabilistic metric spaces and proved a fixed point theorem for mappings in Menger PM spaces with a convex structure. Later, Jesic et al. (2014) have defined a strictly convex, and normal structure in Menger PM spaces. In recent years, fixed point theory in convex PM spaces is developing rapidly. Several fixed point results have been proved for mappings in strictly convex PM spaces. For example see Nikolic et al. (2022), Gabeleh et al. (2025) and Ćirović et al. (2025).

Despite these advances, hyperbolic-type geometric properties of Menger PM spaces remain less explored. Inspired by the concept of hyperbolic metric spaces introduced by W. A. Kirk (1982), this paper focuses on Menger probabilistic metric spaces of hyperbolic type and the study of fixed point results within this probabilistic setting. We examine the structural features of these spaces, present illustrative examples, and prove a fundamental lemma describing the asymptotic behavior of bounded sequences. As a main application, we establish a fixed point theorem for fundamentally nonexpansive mappings acting on nonempty compact convex subsets of Menger PM spaces of hyperbolic type.

This result extends and generalizes a recent fixed point theorem due to Fukhar-Ud-Din (2020), and demonstrates that the combination of PM space with hyperbolic-type properties provides a powerful and flexible tool in fixed point theory.

Preliminaries

We begin by covering essential definitions and mathematical preliminaries that serve as the foundation for deriving our main results.

Definition 1. (Schweizer & Sklar, 1983) A function $F\colon \mathbb{R} \to [0,1]$ is called a *distribution function* if it is nondecreasing, left continuous, and satisfies $\inf_{t\in\mathbb{R}} F(t) = 0$ and $\sup_{t\in\mathbb{R}} F(t) = 1$.

In addition, if $F(0) = 0$, then F is called a *distance distribution function*.

Let D^+ be the set of all distance distribution functions. We denote by ε_0 the maximal element of D^+, which is defined as follows:

$$\varepsilon_0(t) = \begin{cases} 0, & if\ x < 0, \\ 1, & if\ x \geq 0. \end{cases}$$

Definition 2. (Schweizer & Sklar, 1983) A binary operation $T : [0,1] \times [0,1] \to [0,1]$ is a *continuous t-norm* if it is commutative, associative, continuous and such that for all $a, b, c, d \in [0,1]$ it holds that:

(i) $T(a, 1) = a$;
(ii) $T(a, b) \leq T(c, d)$, if $a \leq c$ and $b \leq d$.

Some examples of t-norms include:

$$T_M(a,b) = \min(a,b)\,, T_P(a,b) = a \cdot b, T_L(a,b) = \max\{0, a + b - 1\},$$

$$T_D(a,b) = \begin{cases} \min(a,b)\,, & if \max\{a,b\} = 1 \\ 0, & otherwise \end{cases}.$$

These t-norms satisfy the property: $T_D \leq T_L \leq T_P \leq T_M$. With the exception of T_D, all the aforementioned t-norms are continuous.

Definition 3. (Schweizer & Sklar, 1983) A *Menger probabilistic metric space* (or *Menger PM space*) is a triple $(X, \mathcal{F}, T)$, where: X is a nonempty set, T is a continuous t-norm, and $\mathcal{F}$ is a mapping from $X \times X$ into D^+ such that $\mathcal{F}(x,y) = F_{x,y}$ for every pair $(x,y) \in X \times X$, if and only if the following conditions hold:

(i) $F_{x,y}(t) = \varepsilon_0(t)$ if and only if $x = y$;
(ii) $F_{x,y}(t) = F_{y,x}(t)$, for all $x, y \in X$ and $t \geq 0$;
(iii) $F_{x,z}(t + s) \geq T\left(F_{x,y}(t), F_{y,z}(s)\right)$ for all $x, y, z \in X$ and $t, s \geq 0$.

Remark 1. (Sehgal & Bharucha-Reid, 1972) *Every metric space is a Menger PM space.* Let (X, d) be a metric space and define $T = T_M$ as a continuous t-norm. Also, let $\mathcal{F}(x,y)(t) = \varepsilon_0\big(t - d(x,y)\big)$. Then, $(X, \mathcal{F}, T)$ is a Menger PM space induced by the metric d.

A variety of distinct topological structures can be established on a Menger PM space. Schweizer & Sklar (1983) introduced the (ϵ, λ)-topology by defining a two-parameter family of neighbourhoods N_p of a point $p \in X$ as follows:

$$N_p = \left\{N_p\,(\epsilon, \lambda) \colon \epsilon > 0, \lambda \in (0,1)\right\}$$

where

$$N_P(\epsilon, \lambda) = \{q \in X \colon F_{p,q}(\epsilon) > 1 - \lambda\},$$

The (ϵ, λ)-topology is a Hausdorff topology.

Definition 4. (Schweizer & Sklar, 1983) Let $(X, \mathcal{F}, T)$ be a Menger PM space and let $\{x_n\}$ be a sequence in X. Then:

1) $\{x_n\}$ is said to *converge* to $x \in X$ if, for every $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists a positive integer N such that $F_{x_n,x}(\epsilon) > 1 - \lambda$, whenever $n \geq N$;
2) $\{x_n\}$ is called a *Cauchy sequence* if, for every $\epsilon > 0$ and λ∈(0, 1), there exists a positive integer N such that $F_{x_n,x_m}(\epsilon) > 1 - \lambda$, whenever $n, m \geq N$;
3) A Menger PM space is said to be *complete* if every Cauchy sequence in X converges to a point in X;
4) The function $f: X \to X$ is *continuous* at $x_0 \in X$ if and only if for every sequence $x_n \to x_0$, it holds that $f(x_n) \to f(x_0)$.

Definition 5. (Schweizer & Sklar, 1983) Let $(X, \mathcal{F}, T)$ be a Menger PM space. The *closure* of $A \subset X$ is the union of the set itself with the set of all the limits of sequences from A, and is denoted by $\bar{A}$. (So, $x \in \bar{A}$ if and only if there exists a sequence $\{x_n\} \subset A$ such that $x_n \to x$, for $n \to \infty$). A is said to be *closed*, if it contains all its limit points.

Definition 6. (Schweizer & Sklar, 1983) Let $(X, \mathcal{F}, T)$ be a Menger PM space, then $A \subset X$ is said to be *compact* if every sequence in A has a convergent subsequence.

The concept of probabilistic boundness was introduced by Egbert (1968). A version of this definition follows.

Definition 7. (Egbert, 1968) Let $(X, \mathcal{F}, T)$ be a Menger PM space and $A \subseteq X$. We define the function $\delta_A(t)$ as follows:

$$\delta_A(t) = \sup_{\epsilon < t} \inf_{x,y \in X} F_{x,y}(\epsilon).$$

And call it *the probabilistic diameter of the set A*

If there exists $\lambda \in (0,1)$ such that $\sup_{t>0} \delta_A(t) = 1 - \lambda$, the set A will be called probabilistically semi-bounded. If $\sup_{t>0} \delta_A(t) = 1$, the set A will be called probabilistically bounded.

Remark 2. (Jesic et al., 2014) It is not difficult to see that every metrically bounded set is also probabilistic bounded if it is considered in the induced PM space.

Definition 8. (Hadžić, 1987) Let $(X, \mathcal{F}, T)$ be a Menger PM space. A mapping $W: X \times X \times [0,1] \to X$ is called a *convex* structure on X, if for every $(p,q) \in X \times X$ it holds that:

$$W(p,q,0) = q, \qquad W(p,q,1) = p,$$

and for all $p, q, r \in X,\ \lambda \in (0,1)$, and $t > 0$,

$$F_{W(p,q,\lambda),r}(2t) \geq T\left[F_{p,r}\left(\frac{t}{\lambda}\right), F_{r,q}\left(\frac{t}{1-\lambda}\right)\right].$$

We denote by $(X, \mathcal{F}, T, W)$ a Menger PM space equipped with a convex structure W, and we call it a *convex Menger PM space.*

Definition 9. (Jesic et al., 2014; Gabeleh et al, 2025) A Menger PM space $(X, \mathcal{F}, T)$ is called *metrically convex* if for every $p, q \in X$ and $\lambda \in (0,1)$, there exists a unique element $r \in X$ for which

$$F_{p,r}(t) = F_{p,q}\left(\frac{t}{1-\lambda}\right), \quad F_{r,q}(t) = F_{p,q}\left(\frac{t}{\lambda}\right), \quad \forall t > 0.$$

Now, for $\lambda \in [0,1]$ and $\forall t > 0$, we define:

$$r_\lambda = \begin{cases} q, & \lambda = 0 \\ p, & \lambda = 1 \\ r, & \lambda \in (0,1), \begin{cases} F_{p,r}(t) = F_{p,q}\left(\dfrac{t}{1-\lambda}\right) \\ F_{r,q}(t) = F_{p,q}\left(\dfrac{t}{\lambda}\right) \end{cases} \end{cases}$$

Next, we denote a *metric segment* in a metrically convex Menger PM space by

$$\text{seg}[p,q] = \{r_\lambda, \lambda \in [0,1]\}$$

We now give the following definition:

Definition 10. (Jesic et al., 2014) Let $(X, \mathcal{F}, T)$ be a metrically convex Menger PM space. We will call it *of hyperbolic type* if for all $p, q \in X$ there is a unique $\text{seg}[p,q]$ and for all $p, q, u \in X, r \in \text{seg}[p,q]$, such that for $\lambda \in (0,1)$, and $t > 0$, we have $F_{p,r}(t) = F_{p,q}\left(\frac{t}{1-\lambda}\right), \quad F_{r,q}(t) = F_{p,q}\left(\frac{t}{\lambda}\right), \quad \forall t > 0$, then:

$$F_{u,r}(2t) \geq T\left[F_{u,p}\left(\frac{t}{\lambda}\right), F_{u,q}\left(\frac{t}{1-\lambda}\right)\right].$$

Lemma 1. (Jesic et al., 2014). Let $(X,\mathcal{F},T)$ be a Menger PM space of hyperbolic type. Suppose that for all for $\lambda \in (0,1)$, $t > 0$, $p,q,u \in X$, and $r \in \mathrm{seg}[p,q]$, we have $F_{u,r}(t) > \min\{F_{u,p}(t), F_{u,q}(t)\}$.

If there exists $x \in X$, such that for all $t > 0$:

$$F_{x,r}(t) = \min\{F_{x,p}(t), F_{x,q}(t)\},$$

then $r \in \{p,q\}$.

Example 1. Every convex metric space (M,d) is a metrically convex Menger PM space (M,F,T_M), where $F_{p,q} = \varepsilon_0\,(1 - d(p,q))$.

Example 2. Every normed linear space is a metrically convex Menger PM space.

Example 3. Let $X = \mathbb{R}^+$. For $x,y \in X$, we define $\mathcal{F}(x,y) = F_{x,y}$ such that:

$$F_{x,y}(t) = \begin{cases} 0, & t < 0 \\ \dfrac{t}{t + |x-y|}, & t \geq 0. \end{cases}$$

Also, we consider the Lukasiewicz t-norm defined by:

$$T_L(a,b) = \max\{a+b-1, 0\}, \quad a,b \in [0,1].$$

Then $(\mathbb{R}^+,\mathcal{F},T_L)$ is a Menger PM space of hyperbolic type.

Menger PM space. First, we show that $(\mathbb{R}^+,\mathcal{F},T_L)$ is a Menger PM space of hyperbolic type.

(i) For all $x,y \in \mathbb{R}^+$ and $t \geq 0$:

$F_{x,y}(t) = \varepsilon_0(t) \Leftrightarrow \frac{t}{t+|x-y|} = 1 \Leftrightarrow |x-y| = 0 \Leftrightarrow x = y.$

(ii) Since $|x-y| = |y-x|$, for all $x,y \in \mathbb{R}^+$, then $F_{x,y}(t) = F_{y,x}(t)$.

(iii) For all $x,y,z \in \mathbb{R}^+$, we have $|x-y| \leq |x-z| + |z-y|$. On the other hand for $t,s \geq 0$:

$$F_{x,y}(t+s) = \frac{t+s}{t+s+|x-y|}, \qquad F_{x,z}(t) = \frac{t}{t+|x-z|},$$
$$F_{z,y}(s) = \frac{s}{s+|z-y|},$$

Thus:

$$T_L\left(F_{x,z}(t), F_{z,y}(t)\right) = \frac{t}{t+|x-z|} + \frac{s}{s+|z-y|} - 1,$$

Now we need to prove that:

$$\frac{t+s}{t+s+|x-y|} \geq \frac{t}{t+|x-z|} + \frac{s}{s+|z-y|} - 1 \quad \Leftrightarrow$$
$$1 - \frac{|x-y|}{t+s+|x-y|} \geq 1 - \frac{|x-z|}{t+|x-z|} + 1 - \frac{|z-y|}{s+|z-y|} - 1 \quad \Leftrightarrow$$
$$\frac{|x-y|}{t+s+|x-y|} \leq \frac{|x-z|}{t+|x-z|} + \frac{|z-y|}{s+|z-y|}.$$

The monotony of $f(x) = \frac{x}{k+x}$ implies:

$$\frac{|x-y|}{t+s+|x-y|} \leq \frac{|x-z|+|z-y|}{t+s+|x-z|+|z-y|}$$
$$\leq \frac{|x-z|}{t+s+|x-z|+|z-y|} + \frac{|z-y|}{t+s+|x-z|+|z-y|}$$
$$\leq \frac{|x-z|}{t+|x-z|} + \frac{|z-y|}{s+|z-y|},$$

which proves the Menger PM space triangle inequality. Therefore, $(\mathbb{R}^+, \mathcal{F}, T_L)$ is a Menger PM space.

Metric convexity. Let us show that this space is metrically convex.

(iv) For all $x, y \in \mathbb{R}^+$ and $\lambda \in (0,1)$ there is a unique point

$$z = (1-\lambda)x + \lambda y.$$

Then $|x-z| = (1-\lambda)|x-y|$ and $|z-y| = \lambda|y-x|$.

Consequently, for all $t \geq 0$, we have

$$F_{x,z}(t) = F_{x,y}\left(\frac{t}{1-\lambda}\right), \quad F_{z,y}\left(\frac{t}{\lambda}\right)$$

Thus, $(\mathbb{R}^+, \mathcal{F}, T_L)$ is metrically convex, and for all $x, y \in \mathbb{R}^+$ there exists a unique metric segment $\text{seg}[x, y]$.

Hyperbolic-type condition. Now we show that $(\mathbb{R}^+, \mathcal{F}, T_L)$ is also of hyperbolic type.

(v) Let $u, x, y \in \mathbb{R}^+$ and let $z \in \text{seg}[x, y]$ such that $\forall t \geq 0$, we have

$$F_{x,z}(t) = F_{x,y}\left(\frac{t}{1-\lambda}\right), \quad F_{z,y}\left(\frac{t}{\lambda}\right).$$

For $t \geq 0$, we compute:

$$F_{u,z}(2t) = \frac{2t}{2t+|u-z|},$$

$$F_{u,x}\left(\frac{t}{\lambda}\right) = \frac{\frac{t}{\lambda}}{\frac{t}{\lambda}+|u-x|} = \frac{t}{t+\lambda|u-x|},$$

$$F_{u,y}\left(\frac{t}{1-\lambda}\right) = \frac{\frac{t}{1-\lambda}}{\frac{t}{1-\lambda}+|u-y|} = \frac{t}{t+(1-\lambda)|u-y|}$$

$$T\left(F_{x,y}\left(\frac{t}{\lambda}\right), F_{u,y}\left(\frac{t}{1-\lambda}\right)\right) = \frac{t}{t+\lambda|u-x|} + \frac{t}{t+(1-\lambda)|u-y|} - 1$$

For all $u, x, y \in \mathbb{R}^+$ and $\lambda \in (0,1)$, by the triangle inequality we have:

$$|u-z| \leq \lambda|u-x| + (1-\lambda)|u-y|.$$

Hence, we need to prove:

$$\frac{2t}{2t+|u-z|} \geq \frac{t}{t+\lambda|u-x|} + \frac{t}{t+(1-\lambda)|u-y|} - 1 \quad \Leftrightarrow$$

$$1 - \frac{|u-z|}{2t+|u-z|} \geq 1 - \frac{|u-x|}{t+\lambda|u-x|} + 1 - \frac{|u-y|}{t+(1-\lambda)|u-y|} - 1 \quad \Leftrightarrow$$

$$\frac{|u-z|}{2t+|u-z|} \leq \frac{|u-x|}{t+\lambda|u-x|} + \frac{|u-y|}{t+(1-\lambda)|u-y|} \quad \Leftrightarrow$$

Following the same reasoning as in (iii), we have:

$$\frac{|u-z|}{2t+|u-z|} \leq \frac{\lambda|u-x| + (1-\lambda)|u-y|}{2t+\lambda|u-x|+(1-\lambda)|u-y|}$$
$$\leq \frac{\lambda|u-x|}{t+\lambda|u-x|} + \frac{(1-\lambda)|u-y|}{t+(1-\lambda)|u-y|}.$$

This shows that $(\mathbb{R}^+, \mathcal{F}, T_L)$ is a Menger PM space of hyperbolic type.

Definition 11. Let $(X, \mathcal{F}, T_L)$ be a Menger PM space. The mapping $f: X \to X$ is called nonexpansive if

$$F_{fx,fy}(t) \geq F_{x,y}(t)$$

holds for all $x, y \in X$ and $t > 0$.

Main results

The following Lemma can be viewed as a probabilistic version of a classical Suzuki (2005)-type Lemma in the setting of Menger PM spaces of hyperbolic type.

Lemma 2. (Çobani & Hoxha, in press) Let $(X, \mathcal{F}, T)$ be a Menger PM metric space of hyperbolic type. Let $\{u_n\}$ and $\{v_n\}$ be bounded sequences in X and let $\lambda_n \subset (0,1)$ be such that

$$\lim_{n\to\infty} \lambda_n = 1. \quad (1)$$

If:

$$F_{v_n,u_{n+1}}(t) = F_{v_n,u_n}\left(\frac{t}{1-\lambda_n}\right), \quad F_{u_{n+1},u_n}(t) = F_{v_n,u_n}\left(\frac{t}{\lambda_n}\right), \quad (2)$$

and

$$\lim_{n\to\infty} \sup\left(F_{v_{n+1},v_n}(t) - F_{u_{n+1},u_n}(t)\right) = \varepsilon_0, \quad (3)$$

then

$$\lim_{n\to\infty} F_{v_n,u_n}(t) = \varepsilon_0(t). \quad (4)$$

Proof. Firstly, from conditions (1) and (2) for all $t > 0$, by taking the limit when $n \to \infty$, we get that:

$$\lim_{n\to\infty} F_{v_n,u_{n+1}}(t) = 1, \quad (5)$$
$$\lim_{n\to\infty} F_{u_{n+1},u_n}(t) = F_{v_n,u_n}(t). \quad (6)$$

Next, by applying (6) to the condition (3) of the Lemma, we deduce that for $t > 0$:

$$\lim_{n\to\infty} \sup\left(F_{v_{n+1},v_n}(t) - F_{v_n,u_n}(t)\right) = 0 \text{ and } F_{v_{n+1},v_n}(t) = F_{v_n,u_n}(t). \quad (7)$$

Lastly, since $(X, \mathcal{F}, T)$ is a Menger PM space of hyperbolic type, applying Definition 10, with

$$u = v_{n+1}, \quad r = u_{n+1}, \quad p = v_n, \quad q = u_n,$$

we obtain

$$F_{v_{n+1},u_{n+1}}(2t) \geq T\left[F_{v_{n+1},v_n}\left(\frac{t}{\lambda_n}\right), F_{v_{n+1},u_n}\left(\frac{t}{1-\lambda_n}\right)\right].$$

By taking the limit when $n \to \infty$ and using (7) we have

$$\lim_{n\to\infty} F_{v_{n+1},u_{n+1}}(2t) \geq \lim_{n\to\infty} F_{v_n,u_n}(t), \quad (8)$$

for all $t > 0$. Since the sequences are bounded and (5), (7) hold,

$$\lim_{n\to\infty} F_{v_n,u_n}(t) = 1, \quad t > 0$$

Therefore, $\lim_{n\to\infty} F_{v_n,u_n}(t) = \varepsilon_0(t)$. ■

Definition 12. Let $(X, \mathcal{F}, T)$ be a Menger PM space. The mapping $f: X \to X$ is called fundamentally nonexpansive if:

$$F_{f^2x,f_y}(t) \geq F_{fx,y}(t),$$

Holds for all $x, y \in X$ and $t > 0$

The following theorem is derived from a Fukhar-Ud-Din (2020) theorem adapted to the framework of Menger PM spaces of hyperbolic type.

Theorem 1. Let $(X, \mathcal{F}, T)$ be a Menger PM space of hyperbolic type and let $C \subset X$ be nonempty, compact and convex. If $f: C \to C$ is fundamentally nonexpansive, then f has at least a fixed point in C.

Proof. We begin by choosing an arbitrary point $x_0 \in C$ and we define a sequence $\{fx_n\}$ in C by:

$$F_{fx_n,fx_{n+1}}(t) = F_{fx_n,f^2x_n}\left(\frac{t}{1-\lambda_n}\right), \quad F_{fx_{n+1},f^2x_n}(t) = F_{fx_n,f^2x_n}\left(\frac{t}{\lambda}\right), \quad (9)$$

Where $\lim_{n\to\infty} \lambda_n = 1$.

Since C is convex, we have $f_{x_n} \in C$ for all n.

Because f is fundamentally nonexpansive, it follows that

$$F_{f^2x_{n+1},f^2x_n}(t) \geq F_{fx_{n+1},fx_n}(t), \quad \forall t > 0. \quad (10)$$

Under the assumptions of the theorem, $(X, \mathcal{F}, T)$ is a Menger PM space of hyperbolic type, hence by Lemma 2 by taking $u_n = f^2x_n$ and $v_n = fx_n$ we obtain:

$$\lim_{n\to\infty} F_{f^2x_n,fx_n}(t) = 1, \quad \forall t > 0. \quad (11)$$

Given that C is compact, there exists a subsequence $\{fx_{n_k}\}$ such that

$$fx_{n_k} \to x \in C. \quad (12)$$

By the triangle inequality in Definition 3 (iii), for $t > 0$, we have

$$F_{f^2x_{n_k},x}(t) \geq T\left[F_{f^2x_{n_k},fx_{n_k}}(t-s), F_{fx_{n_k},x}(t)\right],$$

for $t > s > 0$ and letting $k \to \infty$, we obtain

$$\lim_{n\to\infty} F_{f^2x_{n_k},x}(t) = 1$$

Finally, using the Menger PM space triangle inequality in Definition 3 (iii) again,

$$F_{fx,x}(t) \geq T\left[F_{fx,f^2x_{n_k}}(t-s), F_{f^2x_{n_k},x}(t)\right], \text{ for for } t > s > 0. \quad (13)$$

Passing (13) to limit as $k \to \infty$ yields

$$F_{fx,x}(t) = 1, \quad \forall t > 0,$$

which implies that $fx = x$. Hence, x is a fixed point of f in C. ■

Example 4. Consider the Menger PM Space of hyperbolic type $(\mathbb{R}^+, \mathcal{F}, T_L)$ from Example 3. Let $C = [0,3]$ and let $f: C \to C$ be such that $f(x) = \frac{1}{3}x + \frac{1}{2}$.

Hence, we have

$$f^2(x) = \frac{1}{3}f(x) + \frac{1}{2}, \quad f(y) = \frac{1}{3}y + \frac{1}{2}.$$

Consequently:

$$F_{f^2x,fy}(t) = \begin{cases} 0, & t < 0 \\ \dfrac{t}{t + \frac{1}{3}|f(x) - y|}, & t \geq 0 \end{cases},$$

$$F_{fx,y}(t) = \begin{cases} 0, & t < 0 \\ \dfrac{t}{t + |f(x) - y|}, & t \geq 0 \end{cases}.$$

Therefore, for $t \geq 0$

$$\frac{1}{3}|f(x) - y| \leq |f(x) - y| \Leftrightarrow \frac{t}{t + \frac{1}{3}|f(x) - y|} \geq \frac{t}{t + |f(x) - y|}$$

$$\Leftrightarrow F_{f^2x,fy}(t) \geq F_{fx,y}(t),$$

and thus f is fundamentally nonexpansive, and it has a fixed point which is $x = 3$.

Conclusions

In this paper, we outlined the framework of Menger probabilistic metric spaces of hyperbolic type as an extension of Kirk's metric space of hyperbolic type to the probabilistic setting. We established some basic structural properties and provided illustrative examples, including a space constructed on the positive real line and based on the Łukasiewicz t-norm.

A key Suzuki-type lemma describing the asymptotic behavior of bounded sequences in Menger PM spaces of hyperbolic type was proved and used to derive a fixed point theorem for fundamentally nonexpansive mappings on compact convex subsets. These results extend classical fixed point results to the probabilistic setting and contribute to the advancement of nonlinear analysis in Menger probabilistic metric spaces. Future research may focus on obtaining similar results for nonexpansive or quasi-nonexpansive mappings. Moreover, it may be interesting to investigate these results in the context of fuzzy metric spaces.

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